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METHODS OF ESTIMATION FOR MARKETS IN DISEQUILIBRIUM

BY RAY C. FAIR AND DWIGHT M. JAFFEE[1]

This paper is concerned with the econometric problems associated with estimating supply and demand schedules in disequilibrium markets. The general problem is that in the absence of an equilibrium condition the ex ante demand and supply quantities cannot in general be equated to the observed quantity traded in the market. Four methods of estimation, differing primarily in their use of information on price-setting behavior, are developed in this paper. The first method is a generalization of an earlier method developed by R. Quandt and is based upon the maximization of a likelihood function. The method does not require any specific assumption about price-setting behavior, and it allows the sample separation (into demand and supply regimes) to be estimated along with the coefficient estimates. The second and third methods use the change in price as a *qualitative* proxy in determining the sample separation. The fourth method uses the change in price as a *quantitative* proxy for the amount of excess demand (supply) in the market. In the final section of the paper the four methods are used to estimate a model of the housing and mortgage market in an effort to gauge the potential usefulness of each of the methods.

1. INTRODUCTION

BECAUSE OF THE INCREASED concern with the structure of disequilibrium markets,[2] the estimation of supply and demand schedules for such markets has become a problem of practical importance. The main problem of estimation is that in the absence of an equilibrium condition the observed quantity traded in the market may not satisfy both the demand and supply schedules. One general approach to this problem is to try to separate the sample into demand and supply regimes such that each schedule may be appropriately fitted against the observed quantity for the sample points falling within its regime. Another approach is to try to adjust the observed quantity for the effects of rationing and then fit both schedules over the entire sample period using the adjusted quantity.

In this paper four methods of estimating supply and demand schedules in disequilibrium markets are discussed. The first method is a maximum likelihood method for finding the optimal separation of the sample into demand and supply regimes. The other three methods use price-setting information to reduce computational difficulties and to make more use of the available data. The three methods are also based on the assumption that the observed quantity is equal to the

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[2] See, for example, Jaffee [4 and 5] for a discussion of credit rationing in the commercial-loan and mortgage markets and Tucker [9] for the development of a general macroeconomic model of disequilibrium markets.

minimum of the ex ante demand and supply quantities. In Section 2 the four methods are presented, and in Section 3 the methods are used to estimate demand and supply schedules in a model of the housing and mortgage market in an effort to gauge the potential usefulness of each of the methods.

2. FOUR METHODS OF ESTIMATION

The General Model

The general model is assumed to consist of one demand and one supply equation:

$$D_t = \alpha_0 X_t^D + \alpha_1 P_t + \mu_t^D \qquad (t = 1, 2, \ldots, T), \tag{1}$$

and

$$S_t = \beta_0 X_t^S + \beta_1 P_t + \mu_t^S \qquad (t = 1, 2, \ldots, T). \tag{2}$$

D_t denotes the quantity demanded during period t, S_t the quantity supplied during period t, and P_t the price of the good during period t. X_t^D and X_t^S denote the variables, aside from P_t and the error terms μ_t^D and μ_t^S, that influence D_t and S_t respectively.[3] Equations (1) and (2) are standard demand and supply equations: price is assumed to have a negative effect on demand ($\alpha_1 < 0$) and a positive effect on supply ($\beta_1 > 0$).[4] The distinguishing feature of the model is that P_t is *not* assumed to adjust each period so as to equate D_t and S_t.

The Maximum Likelihood Method

Let Q_t denote the actual quantity observed during period t. For the maximum likelihood method it is assumed that Q_t satisfies either the demand schedule or the supply schedule.[5] Under this assumption, equations (1) and (2) can be combined to yield

$$Q_t = k_t(\alpha_0 X_t^D + \alpha_1 P_t + \mu_t^D) + (1 - k_t)(\beta_0 X_t^S + \beta_1 P_t + \mu_t^S) \qquad (t = 1, 2, \ldots, T), \tag{3}$$

where

$$k_t = \begin{cases} 0 & \text{if} \quad Q_t = S_t, \\ 1 & \text{if} \quad Q_t = D_t. \end{cases}$$

[3] Generally, X_t^D and X_t^S should be considered to be vectors of variables, with α_0 and β_0 being corresponding vectors of coefficients, but without loss of generality in the following analysis X_t^D and X_t^S will be taken to be single variables.

[4] More rigorously, P_t is meant to refer to the relative price of the good, i.e., the price of the good normalized by some general price index. Also, it is not necessary that the current value of P appear in equations (1) and (2). D_t and S_t may respond to some lagged value of P, and indeed the lag may even be different in the two equations.

[5] In practice, of course, it may be the case that the observed quantity satisfies neither the demand schedule nor the supply schedule. In this case, some assumption would have to be made about how the observed quantity is determined before the supply and demand schedules could be estimated. All of the methods discussed in this paper rely on the assumption that the observed quantity satisfies either the demand schedule or the supply schedule.

The general problem in (3) is to estimate the values of the parameters α_0, α_1, β_0, and β_1, and the T values of k_t, given observations on X_t^D, X_t^S, P_t, and Q_t ($t = 1, 2, \ldots, T$). The problem can be looked at, in other words, as one of choosing those observations (sample points) for which $Q_t = D_t$ ($k_t = 1$) and those for which $Q_t = S_t$ ($k_t = 0$) and then estimating the parameters of each equation by some standard technique, such as ordinary least squares, over the relevant sample points. Note that when $D_t = S_t$ ($=Q_t$), k_t in (3) is indeterminant. If this occurs for some value of t, k_t must be arbitrarily assumed equal to zero or one.

It will also be assumed that the error terms μ_t^D and μ_t^S are normally and independently distributed and are independent of X_t^D, X_t^S, and P_t.[6] An appropriate method of estimation of the parameters in equation (3) can then be seen to be a generalization of Quandt's [**8**] maximum likelihood technique for estimating the position of a switching point in a linear regression system. Quandt was concerned with determining one switching point in a regression system obeying two regimes, whereas in the more general case here the concern is with determining a potentially large number of switching points. A switching point occurs each time the situation changes from the quantity demanded being observed to the quantity supplied being observed or vice versa. For the present case the likelihoods of the supply and demand observations are:

$$(2\pi\sigma_D^2)^{-m/2} \exp\left[-\frac{1}{2\sigma_D^2}\sum_t^m (D_t - \alpha_0 X_t^D - \alpha_1 P_t)^2\right] \tag{4}$$

and

$$(2\pi\sigma_S^2)^{-n/2} \exp\left[-\frac{1}{2\sigma_S^2}\sum_t^n (S_t - \beta_0 X_t^S - \beta_1 P_t)^2\right], \tag{5}$$

where m is the number of observations for which $Q_t = D_t$ (i.e., for which $k_t = 1$), and n is the number of observations for which $Q_t = S_t$ (i.e., for which $k_t = 0$).[7] The sums Σ_t^m and Σ_t^n denote summation over those observations (not necessarily sequential) for which $Q_t = D_t$ and $Q_t = S_t$ respectively. The terms σ_D and σ_S are the standard deviations of μ_t^D and μ_t^S respectively. Using (4) and (5), the likelihood function of the entire sample is:

$$\begin{aligned} L = (2\pi\sigma_D^2)^{-m/2}(2\pi\sigma_S^2)^{-n/2} \exp\Bigg[&-\frac{1}{2\sigma_D^2}\sum_t^m (D_t - \alpha_0 X_t^D - \alpha_1 P_t)^2 \\ &-\frac{1}{2\sigma_S^2}\sum_t^n (S_t - \beta_0 X_t^S - \beta_1 P_t)^2\Bigg]. \end{aligned} \tag{6}$$

[6] It may, of course, not be appropriate to assume that the error terms and P_t are independent. In the standard equilibrium case the error terms and P_t are correlated, and even for the non-equilibrium case if P_t is determined by an equation like (11) below, the error terms and P_t will be correlated. Notice, however, that if some lagged value of P is used in equations (1) and (2) instead of the current value, the correlation problem is less likely to arise. Whatever the case, it must be assumed that all of the explanatory variables in equations (1) and (2) are uncorrelated with the error terms in the two equations if the following maximum likelihood technique is to be valid.

[7] Note that m plus n equals T, where T is the total number of observations on Q_t.

For a given sample separation (i.e., for given values of k_t in (3)), the log of L in (6) can be differentiated with respect to $\alpha_0, \alpha_1, \beta_0$, and β_1. Setting these derivatives equal to zero and solving the resulting equations yields the normal least squares estimates for the four coefficients, $\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_0$, and $\hat{\beta}_1$.[8] Setting the partial derivatives of $\log L$ with respect to σ_D and σ_S equal to zero and using the values of $\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_0$, and $\hat{\beta}_1$ yields:

$$\text{(7)} \qquad \hat{\sigma}_D^2 = \frac{\sum_t^m (D_t - \hat{\alpha}_0 X_t^D - \hat{\alpha}_1 P_t)^2}{m},$$

and

$$\text{(8)} \qquad \hat{\sigma}_S^2 = \frac{\sum_t^n (S_t - \hat{\beta}_0 X_t^S - \hat{\beta}_1 P_t)^2}{n}.$$

Finally, substituting $\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}_D^2$, and $\hat{\sigma}_S^2$ into the log of (6) yields

$$\text{(9)} \qquad \log L = -(m+n)\log\sqrt{2\pi} - m\log\hat{\sigma}_D - n\log\hat{\sigma}_S - \frac{m+n}{2}.$$

The solution to the problem is then to choose the sample separation and the corresponding values of $\hat{\alpha}_0$, $\hat{\alpha}_1$, $\hat{\beta}_0$,$\hat{\beta}_1$ that maximize $\log L$ in (9). Quandt for his single switching point case has suggested that (9) be maximized by calculating $\log L$ for all possible values of the switching point and selecting that point for which the computed value of $\log L$ is the largest. The analogous suggestion here is to calculate $\log L$ for all possible pairs of supply and demand sample periods and to choose that pair for which $\log L$ is at a maximum. The estimates for $\alpha_0, \alpha_1, \beta_0$, β_1 would then be the least squares estimates over the respective sample periods. For Quandt's case, T calculations are needed to find the maximum, but in the more general case here, 2^T calculations are necessary.[9] The question of whether there are algorithms available that can be used to find the maximum of $\log L$ is discussed in Section 3.

Assuming that the maximum of the likelihood function can be found, the estimates are consistent. An elegant proof of consistency of maximum likelihood estimators that requires only very weak regularity conditions on the likelihood function has been given by Kendall and Stuart [**6**, pp. 39–41]. The most restrictive condition in the proof is that the expected value of the log of the likelihood function must exist when the true values of the parameters hold. It is easy to show that this condition holds for the function in (6). For the true sample separation and param-

[8] In the normal least squares equations the summations for $\hat{\alpha}_0$ and $\hat{\alpha}_1$ are over the m demand observations and the summations for $\hat{\beta}_0$ and $\hat{\beta}_1$ are over the n supply observations.

[9] The 2^T figure is derived by noting that there are two ways in which the first sample point can be chosen (either $Q_t = D_t$ or $Q_t = S_t$), two ways in which the second sample point can be chosen, and so on through all T sample points. In finite samples, $\log L$ has a corner solution for m or n equal to 2. In practice this problem can be avoided by requiring that some minimum number of observations be left in any one equation. This means that the actual number of calculations required to find the maximum is slightly less than 2^T.

eter values, the expected value of the log of (6) is merely the sum of the expected values of the logs of the likelihood functions of the separate supply and demand sample points, and the latter two expected values are known to exist from standard regression theory.[10]

One further property of the maximum likelihood technique should be emphasized. The technique assumes that any given sample point can be classified either as a demand observation or as a supply observation, but not both. It is not possible, in other words, to specify using prior information the existence of periods of equilibrium.

Additional Assumptions

Although the maximum likelihood method has the advantage of being based on fairly weak assumptions, it may be the case for many economic problems that additional information about market behavior is available. It seems reasonable, for example, to expect that if the observed quantity is equal to either the quantity demanded or the quantity supplied, it will be equal to the *minimum* of these two quantities:

$$(10) \qquad Q_t = \min\{D_t, S_t\} \qquad (t = 1, 2, \ldots, T).$$

In other words, it seems reasonable to expect that if the quantity demanded exceeds the quantity supplied, demanders will go unsatisfied, and if the quantity supplied exceeds the quantity demanded, suppliers will go unsatisfied. The following three methods are based on the assumption in (10).

Additional information about price-setting behavior may also be available to help in the estimation problem. Most dynamic theories of price-setting behavior formulate the change in price as some function of the excess demand existing in the market. If the change in price and excess demand are related in this manner, then the change in price may be used as an indicator of the amount of excess demand (or supply) in the market. In particular, it may be possible to postulate the following price-adjustment equation:

$$(11) \qquad \Delta P_t = f[D_t - S_t], \qquad f'[D_t - S_t] > 0,$$

where the price change is assumed to be a positive function of the excess demand in the market.[11] More specifically, it may be possible to postulate that

$$(12) \qquad \Delta P_t \gtreqless 0 \quad \text{as} \quad D_t - S_t \gtreqless 0,$$

[10] It should be noted, however, that the standard errors of the sample separations are not well defined because the likelihood function is not continuous in the k_t coefficients. Furthermore, because the likelihood function is not continuous, one cannot rely on the standard asymptotic properties of maximum likelihood estimators, which are based on the assumption that the likelihood function is continuous and differentiable. If, in practice, the standard errors of the estimates of the α and β coefficients are calculated for a given sample separation, the standard errors are conditional on this separation.

[11] In this context it should be stressed, as noted in Footnote 4, that P_t is to be interpreted as the relative price of the good, i.e., the price of the good deflated by some general price index.

or to postulate that

(13) $\Delta P_t = \gamma(D_t - S_t), \qquad 0 \leqslant \gamma \leqslant \infty.$

Equation (12) implies that the *direction* of the price change is an indicator of the excess demand *status* of the market. This assumption will be used in deriving the second and third methods of estimation (to be denoted as directional methods I and II). Equation (13) implies that the *amount* of the price change is directly proportional to the *amount* of excess demand. The coefficient of proportionality, γ, depends on the length of the time unit,[12] where γ equal to zero is the polar case of no adjustment, and γ equal to infinity is the polar case of perfect adjustment.[13] This second assumption will be used in deriving the fourth method of estimation (to be denoted the quantitative method).

Directional Method I

Directional method I can be illustrated by graphing the demand and supply functions (1) and (2) against the price, as shown in Figure 1. The market clearing price is shown as P^*. Whenever the quoted price is less than P^* there is excess demand, which implies from equation (12) that the price must be rising. Furthermore, from equation (10) it is known that in periods of excess demand, supply will equal the observed quantity, while demand will be unobserved. Conversely, whenever the quoted price is greater than P^* there is excess supply, which means that the price must be falling. It is also implied from equation (10) that in periods of excess supply, demand will equal the observed quantity, while supply will be unobserved. Consequently, in periods of rising prices only the supply schedule will be observed and in periods of falling prices only the demand schedule will be observed.

To implement directional method I, one first separates the sample into periods of excess demand and excess supply on the basis of the observed price change. The supply function can be estimated over periods of excess demand (using Q as the dependent variable), and the demand function can be estimated over periods

[12] Since the formulation of equation (13) is in discrete time units, D_t and S_t should be interpreted as averages over the time intervals. The magnitude of γ will thus vary depending on the length of the time interval. This can be seen more formally by specifying equation (13) in continuous time as $dP_t/dt = \gamma(D_t - S_t)$. Integrating both sides from t_0 to t_1 yields:

$$P_{t_1} - P_{t_0} = \gamma \int_{t_0}^{t_1} (D_t - S_t)\, dt = \gamma(t_1 - t_0) \int_{t_0}^{t_1} [(D_t - S_t)/(t_1 - t_0)]\, dt.$$

The integral on the right hand side is then the average excess demand over the time interval, and the proportionality factor is the instantaneous rate of adjustment corrected for the length of the time interval.

[13] It should be stressed that the polar case of perfect adjustment corresponds to the market always being in equilibrium. The assumption that the market is always in equilibrium can be tested in the work below by testing whether the estimate of $1/\gamma$ is significantly different from zero. If the estimate is not significantly different from zero, then the null hypothesis of perfect market adjustment cannot be rejected.

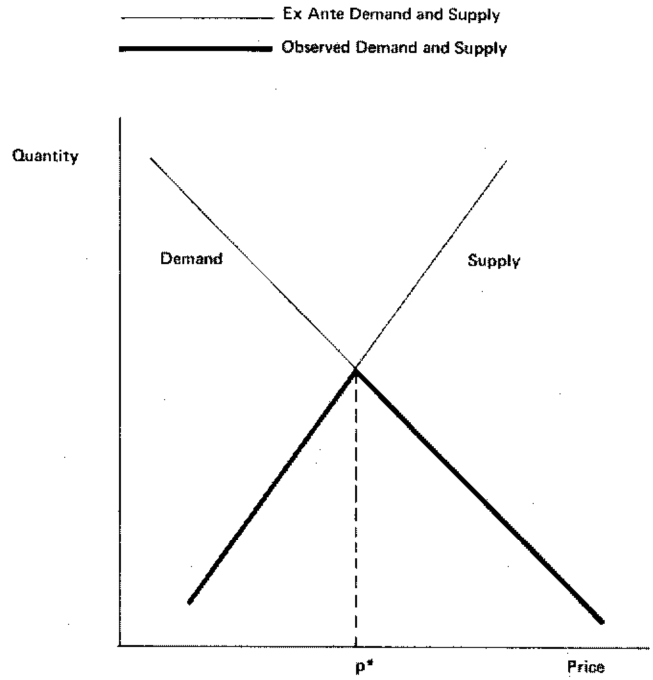

FIGURE 1.

of excess supply (also using Q as the dependent variable).[14] Periods of temporary equilibrium ($\Delta P_t = 0$) are included in both samples since both schedules are observed at such times. In terms of Figure 1 it can be seen that only the darkened portions of the two functions will be directly estimated with this method.

The sample separation implied by directional method I will be correct under the assumptions of the model as long as no error occurs in the price equation (equation (12)). There is, however, a source of inconsistency that arises even when the sample separation is correct.[15] Consider, for example, the demand equation. Q_t will be equal to D_t (and thus the demand equation estimated) whenever D_t is less than or equal to S_t, or whenever $\alpha_0 X_t^D + \alpha_1 P_t + \mu_t^D$ is less than or equal to $\beta_0 X_t^S + \beta_1 P_t + \mu_t^S$. Now, other things being equal, D_t will be less than S_t more often when μ_t^D is small than when it is large. The mean of μ_t^D *over those points for*

[14] The addition of equation (12) to the model makes P_t an endogenous variable, and this will be the source of simultaneous equation bias if the current value of P is specified in equations (1) and (2). The bias can be avoided by using a technique such as two-stage least squares. If the two-stage least squares technique is used, however, it should be remembered that when estimating the supply equation, the first stage regression of P_t against the instruments must be over the supply sample period, and when estimating the demand equation, the first stage regression must be over the demand sample period. Otherwise, in the second stage regression for each equation, the orthogonality between the predicted value series and the reduced form error term is not preserved. McCarthy [7] presents a formal proof that this orthogonality must be preserved to insure consistent second-stage estimates.

[15] The authors are indebted to Harry H. Kelejian for this point.

which demand is observed will thus not be zero. More seriously, the mean of μ_t^D over those points for which demand is observed will not be independent of X_t^D and P_t. Large values of X_t^D, for example, will less often be associated with large values of μ_t^D than with small values of μ_t^D over those points for which demand is observed. Similar considerations apply for the supply equation. The estimates under directional method I are thus not consistent even though the sample separation is correct.

Directional Method II

Directional method II is less dependent on the price change as an indicator of excess supply or demand in the market. Equation (12), for example, may not hold exactly: there may be periods in which the change in price is either so small or so variable as to leave the excess demand status of the market in doubt. Also, the price may respond to some lagged value of excess demand rather than to the current value and thus leave doubt as to the excess demand status of the market at turning points in the price change series.[16] Directional method II operates by postulating a number of different sample separations corresponding to alternative assumptions about the excess demand status of the market during the "doubtful" periods. The supply and demand equations are then fitted to each set of sample separations and the likelihood function (9) evaluated for each case. The preferred sample separation is then taken to be the one that maximizes (9).

Directional method II can thus be considered to be a version of the likelihood technique. Prior information on price changes is merely used to reduce the number of sample separations over which direct evaluation of the likelihood function is necessary. Note that since directional method II relies on the evaluation of the likelihood function, demand and supply sample periods cannot overlap when the method is used. Sample points must either be put into the demand regime or the supply regime, but not both. This differs from directional method I, which assumes that any sample points for which ΔP_t equals zero are equilibrium points and thus belong in both regimes. Directional method I, of course, does not rely on the evaluation of the likelihood function for the sample separation.

The Quantitative Method

For the quantitative method, the adjustment process is assumed to take the form of equation (13). Solving this equation for the excess demand yields the following:

$$(14) \qquad D_t - S_t = \frac{1}{\gamma}(\Delta P_t).$$

[16] The simplest case is when the price responds to excess demand with a single discrete lag, say of i periods. In this case the change in price i periods in the future should be used as the indicator of current excess demand. A more complicated generalization might assume that the price change is a distributed lag of past excess demands. A distributed "lead" of price changes could then be used to indicate the current excess demand status.

If γ can be estimated, then the actual amount of excess demand can be directly determined from the change in price, and thus both the demand and supply schedules can be estimated over the entire sample period. The procedure to be developed here simultaneously estimates γ and the parameters of the two schedules.

First, consider a period with rising prices. From equation (14) it is known that this will be a period of excess demand, and thus (from equation (10)) the observed quantity will equal the supply. Consequently the supply function can be directly estimated using the observed quantity as the dependent variable:

$$Q_t = S_t = \beta_0 X_t^S + \beta_1 P_t + \mu_t^S, \qquad \Delta P_t \geqslant 0. \tag{15}$$

Furthermore, because the supply equals the observed quantity, equation (14) can be rewritten as:

$$Q_t = D_t - \frac{1}{\gamma}\Delta P_t = \alpha_0 X_t^D + \alpha_1 P_t - \frac{1}{\gamma}\Delta P_t + \mu_t^D, \qquad \Delta P_t \geqslant 0. \tag{16}$$

Thus the parameters of the demand function can also be estimated, using the observed quantity as the dependent variable, as long as the change in price is included in the equation as an implicit adjustment for the amount of rationing.

In periods of falling prices essentially the same principles apply. The supply and demand functions will then be estimated as, respectively:

$$Q_t = S_t - \frac{1}{\gamma}|\Delta P_t| = \beta_0 X_t^S + \beta_1 P_t - \frac{1}{\gamma}|\Delta P_t| + \mu_t^S, \qquad \Delta P_t \leqslant 0, \tag{17}$$

and

$$Q_t = D_t = \alpha_0 X_t^D + \alpha_1 P_t + \mu_t^D, \qquad \Delta P_t \leqslant 0. \tag{18}$$

Indeed, the system of equations (15)–(18) can be reduced to a single demand equation and a single supply equation, each to be estimated over the entire sample period, by making the appropriate adjustment for the change in price:

$$Q_t = D_t - \frac{1}{\gamma}/\Delta P_t/ = \alpha_0 X^D + \alpha_1 P_t - \frac{1}{\gamma}/\Delta P_t/ + \mu_t^D, \tag{19}$$

where

$$/\Delta P_t/ = \begin{cases} \Delta P_t & \text{if } \Delta P_t \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Q_t = S_t - \frac{1}{\gamma}\backslash\Delta P_t\backslash = \beta_0 X_t^S + \beta_1 P_t - \frac{1}{\gamma}\backslash\Delta P_t\backslash + \mu_t^S, \tag{20}$$

where

$$\backslash\Delta P_t\backslash = \begin{cases} -\Delta P_t & \text{if } \Delta P_t \leqslant 0; \\ 0 & \text{otherwise.} \end{cases}$$

It is apparent that equation (19) is equivalent to the two demand equations (16) and (18) and that equation (20) is equivalent to the two supply equations (15) and (17).

Two problems occur in the estimation of equations (19) and (20). One problem is that the same coefficient $1/\gamma$ appears in both equations. This constraint can be taken into account by using the estimation technique developed in Fair [2]. The second problem is the possibility of simultaneous equation bias due to the endogeneity of P_t, $\diagup \Delta P_t \diagup$, and $\diagdown \Delta P_t \diagdown$.

The problem of simultaneous equation bias can be handled by the standard two-stage least squares technique, but the step function characteristic of $\diagup \Delta P_t \diagup$ and $\diagdown \Delta P_t \diagdown$ makes the application of the technique somewhat more complicated than usual. To simplify, start by assuming that the level of P enters equations (19) and (20) only with a lag, so that the only possible source of simultaneity bias is through $\diagup \Delta P_t \diagup$ and $\diagdown \Delta P_t \diagdown$. In applying the two-stage least squares technique to one of the equations, say (19), the standard procedure is to replace $\diagup \Delta P_t \diagup$ with fitted values (denoted as $\diagup \Delta \hat{P}_t \diagup$) obtained by regressing the variable against the appropriate instruments. The difficulty in the present case is that, by definition, $\diagup \Delta P_t \diagup$ must be zero whenever ΔP_t is negative, so that $\diagup \Delta \hat{P}_t \diagup$ cannot be obtained by estimating the first stage regression over the entire sample period and using the entire set of fitted values. A consistent procedure is to estimate the first stage regression only over that part of the sample for which ΔP_t is non-negative and use the fitted values from the regression to construct the $\diagup \Delta \hat{P}_t \diagup$ series, with zeros inserted for the periods of negative ΔP_t. The endogeneity of $\diagdown \Delta P_t \diagdown$ in equation (20) can be treated in a symmetrical manner, with the first-stage regression estimated only over that part of the sample for which ΔP_t is non-positive.[17]

Now consider the case in which P_t, as well as $\diagup \Delta P_t \diagup$ and $\diagdown \Delta P_t \diagdown$, is endogenous. The $\hat{P}_t$ series must be constructed as $\diagup \Delta \hat{P}_t + \diagdown \Delta \hat{P}_t \diagdown + P_{t-1}$, where $\diagup \Delta \hat{P}_t \diagup$ and $\diagdown \Delta \hat{P}_t \diagdown$ are obtained as above. The $\hat{P}_t$ series cannot be obtained from the first stage regression of P_t (or ΔP_t)[18] over the entire sample period because this will not preserve in the second stage regression for each equation the orthogonality between the two fitted value series and the reduced form error terms (see Footnotes 14 and 17). It should also be noted that in the above construction of the $\hat{P}_t$ series, $\diagup \Delta \hat{P}_t \diagup$ and $\diagdown \Delta \hat{P}_t \diagdown$ have to be obtained from first stage regressions which are estimated

[17] Alternatively, consistent estimates of the coefficients of equation (19) can be obtained by estimating the first-stage regression over the sample corresponding only to *positive* values of ΔP_t, with zeros inserted in $\diagup \Delta \hat{P}_t \diagup$ for periods of non-positive values ΔP_t. Likewise, consistent estimates of the coefficients of equation (20) can be obtained by estimating the first stage regression over the sample corresponding only to *negative* values of ΔP_t, with zeros inserted in $\diagdown \Delta \hat{P}_t \diagdown$ for periods of non-negative values of ΔP_t. It should be noted that consistent estimates of the coefficients of, say, equation (19) cannot be obtained by estimating the first stage regression over the entire sample period and using the fitted values from this regression to construct the $\diagup \Delta \hat{P}_t \diagup$ series, with zeros inserted for periods of negative (or non-positive) values of ΔP_t. It is easy to see that this procedure does not preserve in the second-stage regression the orthogonality between the fitted value series and the reduced form error term. As mentioned in Footnote 14, this orthogonality is necessary to insure consistent estimates (see also McCarthy [7]).

[18] Since P_{t-1} should be used as an instrument in the first stage regression, it makes no difference whether P_t or ΔP_t is used as the dependent variable in the first-stage regression.

over mutually exclusive (non-overlapping) sample periods. Otherwise it can be shown that the orthogonality property is not preserved.

This two-stage least squares solution to the problem of simultaneous equation bias does not, of course, take into account the constraint that the same coefficient appears in both equations. Techniques for dealing with both constraints across equations and simultaneous equation bias at the same time are not well developed. For present purposes, it is not clear whether it is better to use the two-stage least squares technique to obtain consistency at a cost of some efficiency or whether it is better to impose the constraint and gain efficiency at a cost of consistency. In the example below both of these techniques are used, and the difference in results for this example can be compared.

3. AN EXAMPLE FROM THE HOUSING SECTOR

The Model

The housing market appears to be a good example of a market which is not always in equilibrium and in which rationing does occur. The basic housing model used here was developed in Fair [**1**] within the context of an aggregate forecasting model. Because the model was designed to be used for forecasting purposes, it was kept fairly simple, and this has obvious advantages for purposes of this paper as well. The outline of the model is as follows.[19] The demand for housing starts during month t, HS_t^D, is assumed to be a function of (i) population growth and trend income, both of which are approximated by a time trend, (ii) the stock of houses in existence or under construction during the previous month, and (iii) the mortgage rate lagged two months, RM_{t-2}. The stock of houses in existence or under construction is approximated by the sum (from an arbitrary base period value) of past housing starts less a trend approximation for past removals. The basic equation for HS_t^D is thus:

$$(21) \qquad HS_t^D = \alpha_0 + \alpha_1 t + \alpha_2 \sum_{i=1}^{t-1} HS_i + \alpha_3 RM_{t-2} + \mu_t^D.$$

The summation of HS_i in (21) (HS_i is the actual number of housing starts during month i) arbitrarily assumes that the initial stock of houses in month 0 is zero, and this error is absorbed in the constant term α_0 in the equation. The time trend t in (21) is picking up both the population growth-trend income effect as well as the effect of past removals. The mortgage rate, RM, is the price variable in the model, and the best results were achieved by lagging the rate two months in equation (21).

The supply of housing starts during month t, HS_t^S, is assumed to be a function of lagged deposit flows into savings and loan associations (SLAs) and mutual

[19] The following outline of the model is quite brief, since it is not the purpose of this paper to provide an elaborate defense of the particular example used. In particular, the outline in this paper ignores the distinction between mortgage funds for housing starts and the housing starts themselves, and it ignores the construction side of the housing market altogether. A complete discussion of the housing starts model, including these latter two issues, is presented in Fair [**1**, Chapter 8].

savings banks (MSBs) and of the mortgage rate lagged one month. Let DF_t denote the flow of private deposits into SLAs and MSBs during month t and let DHF_t denote the flow of borrowings by the SLAs from the Federal Home Loan Bank (FHLB) during month t. Then the two flow variables which have been included in the supply equation are the six-month moving average of DF_t lagged one month (denoted as $DF6_{t-1}$) and the three-month moving average of DHF_t lagged two months (denoted as $DHF3_{t-2}$).[20] The equation for HS_t^S is thus:

$$(22) \qquad HS_t^S = \psi_0 + \psi_1 t + \psi_2 DF6_{t-1} + \psi_3 DHF3_{t-2} + \psi_4 RM_{t-1} + \mu_t^S.$$

Since the housing starts variable is in units of starts and since the deposit flow variables are in current dollar terms, a time trend is added to (22) to pick up the possible trend increase in the deposit-housing starts ratio due to an increase in the dollar value of each start. The mortgage rate is added to (22) on the grounds that for a given flow of deposits, SLAs and MSBs are likely to put more of this flow into the mortgage market the higher is the rate of return on mortgages. Also, other funds which do not go through the SLAs and MSBs may be more attracted to the mortgage market the higher is the mortgage rate.[21] The best results were achieved by lagging the mortgage rate one month in equation (22).

Finally, the actual number of housing starts, HS_t, is assumed to be determined by an equation like (10):

$$(23) \qquad HS_t = \min\{HS_t^D, HS_t^S\}.$$

It is assumed, in other words, that the housing market is not necessarily in equilibrium.

The model to be estimated thus consists of equations (21), (22), and (23). Since RM is the price variable in the model, when the quantitative method is used, the equation for ΔRM_t is:

$$(24) \qquad \Delta RM_t = \gamma(HS_t^D - HS_t^S).$$

The Results

Before presenting the results, mention should be made of the properties assumed about the error terms μ_t^D and μ_t^S in equations (21) and (22). The errors were assumed to be first order serially correlated—$\mu_t^D = \rho_D \mu_{t-1}^D + \varepsilon_t^D$ and $\mu_t^S = \rho_S \mu_{t-1}^S +$

[20] Different lags and moving averages of DF and DHF were tried in the initial estimates of the model, and $DF6_{t-1}$ and $DHF3_{t-2}$ appeared to give the best results. The results were not very sensitive to slightly different specifications.

[21] Theoretically, it is not the absolute size of the mortgage rate that should matter, but the size of the mortgage rate relative to rates on alternative assets. In the initial estimates of the model various yield differential variables were tried, but with no success. While theoretically not very satisfying, it appeared to be the absolute level of rates which mattered and not rate differences.

Various deposit and mortgage stock variables were also tried in equation (22) with no real success. The flow variables always dominated the stock variables, which might indicate that the adjustment of SLAs and MSBs to changing deposit conditions is fairly rapid.

TABLE I

ESTIMATES OF THE DEMAND AND SUPPLY EQUATIONS USING THE VARIOUS METHODS[a]

		Demand								
		Constant	t	$\sum_{i=1}^{t-1} HS_i$	RM_{t-2}	$/\Delta RM_t/$	$\hat{\rho}_D$	SE	R^2	Number of Observations
(1)[b]	(D-1)	120.92 (49.56)	8.72 (3.81)	−.073 (0.32)	−.139 (.088)		.830	9.23	.891	126
(2)	(D-2)	193.16 (62.31)	6.78 (3.37)	−.055 (.028)	−.241 (.106)		.731	8.74	.915	85
(3)	(D-3)	328.43 (54.20)	3.94 (2.33)	−.032 (.020)	−.471 (.082)		.499	8.09	.943	62
(4)	(D-4)	119.94 (49.98)	8.48 (3.87)	−.071 (.033)	−.137 (.089)	−.404 (.168)	.839	9.04	.897	126
(5)	(D-5)	119.84 (47.00)	8.43 (3.65)	−.071 (.031)	−.136 (.083)	−.408 (.147)	.839	9.04	.897	126
(6)	(D-6)	119.91 (52.36)	8.44 (4.12)	−.070 (.035)	−.137 (.094)	−.546 (.167)	.850	9.07	.896	126

		Supply									
		Constant	t	$DF6_{t-1}$	$DHF3_{t-2}$	RM_{t-1}	$\backslash \Delta RM_t \backslash$	$\hat{\rho}_S$	SE	R^2	Number of Observations
(1)	(S-1)	−45.10 (31.54)	−.158 (.069)	.054 (.007)	.050 (.010)	.093 (.042)		.521	8.34	.912	126
(2)	(S-2)	−40.84 (31.66)	−.236 (.076)	.048 (.008)	.033 (.012)	.116 (.043)		.574	7.59	.925	108
(3)	(S-3)	−75.87 (43.60)	−.332 (.123)	.047 (.011)	.012 (.019)	.190 (.069)		.697	6.86	.934	64
(4)	(S-4)	−45.51 (31.17)	−.163 (.068)	.054 (.007)	.049 (.010)	.095 (.041)	−.448 (.477)	.510	8.35	.913	126
(5)	(S-5)	−45.46 (28.96)	−.162 (.063)	.054 (.007)	.049 (.010)	.095 (.038)	−.408 (.147)	.512	8.35	.913	126
(6)	(S-6)	−44.45 (34.19)	−.183 (.075)	.054 (.008)	.036 (.011)	.107 (.045)	−1.112 (.520)	.510	9.15	.896	126

[a] Dependent variable is HS_t; standard errors are in parentheses; $\hat{\rho}_D$ and $\hat{\rho}_S$ are the estimates of the first order serial correlation coefficients.

[b] The various methods, numbered (1)–(6), are as follows: (1) assumption that demand and supply are always equal; (2) directional method I; (3) directional method II; (4) quantitative method, no constraint imposed, no account for simultaneity bias; (5) quantitative method, constraint imposed, no account for simultaneity bias; and (6) quantitative method, no constraint imposed, account for simultaneity bias.

ε_t^S—where ε_t^D and ε_t^S are assumed to be normally and independently distributed with zero means and constant variances and to be uncorrelated with their own past values. How this assumption affects the use of each of the techniques will be discussed below.

To have a basis of comparison, equations (21) and (22) were first estimated under the assumption that HS_t^D and HS_t^S are always observed (and equal to HS_t).

The results are presented in lines (D-1) and (S-1) of Table I.[22] Looking at the demand equation, the estimates of the coefficients of the housing stock variable and the mortgage rate variable are negative, as expected. The estimate of the coefficient of the time trend is positive, which also is as expected since the time trend is mainly proxying for population growth and trend income. The coefficient estimates in the supply equation are also of the expected sign. Contrary to the situation in the demand equation, the time trend has a negative effect and the mortgage rate a positive effect in the supply equation. The two deposit flow variables have a positive effect, as expected.

An attempt was next made to estimate equations (21) and (22) by the maximum likelihood technique.[23] Since scanning over all 2^T possibilities was clearly not feasible (T in this case was 126), a number of algorithms were tried. Unfortunately, none of the attempts met with any success. For even slightly different starting points the algorithms led to different solutions, and there was no evidence that any of them was leading to a global maximum. The general conclusion seemed to be that the logarithm of the likelihood function (as in (9)) was not well behaved enough to allow a global solution to be found short of essentially scanning over all 2^T possibilities. While one should not generalize too much from one example, the results achieved in this study are not encouraging as to the usefulness of the maximum likelihood technique. Unless better algorithms than those considered in this study can be found or unless the likelihood function is better behaved for other examples, the maximum likelihood technique does not appear to be of much practical use.

[22] The equations were estimated by the Cochrane-Orcutt iterative technique for the period June, 1959 to November, 1969. None of the data were seasonally adjusted. The series used for HS is the series on private non-farm housing starts; the series used for RM is the FHA mortgage rate series on new homes; the SLA part of DF is the change in savings capital of SLAs and the MSB part of DF is the change in deposits of MSBs; and the series used for DHF is the change in total advances of the Federal Home Loan Bank. Since none of the data were seasonally adjusted, eleven dummy variables were added to each equation. Also, in an effort to adjust for the number of work days in the month a monthly "work day" variable was constructed and was added to each equation. The construction of this variable and the need for its inclusion are explained in Fair [1]. To economize on space, the coefficient estimates of the dummy variables and the work day variable are not presented in Table I.

[23] For most of the experimentation with this technique, the variables were seasonally adjusted and the dummy variables were dropped from the equation. This was done primarily to minimize computational time. The coefficient estimates achieved using seasonally adjusted variables were quite close to those achieved using seasonally unadjusted and dummy variables. The serial correlation assumption posed no particular programming difficulties, as iterative procedures could be easily used, but in order to use the technique an additional assumption had to be made. Assume, for example, that for point $t-1$, D_{t-1} is equal to the observed Q_{t-1}, and that for point t, S_t is equal to Q_t. Because of the serial correlation assumption, S_{t-1} enters as an explanatory variable for S_t (and likewise D_{t-1} as an explanatory variable for D_t). Then, strictly speaking, point t cannot be used to estimate the supply equation in this example, because S_{t-1} is not observed. The assumption was thus made that for the month before a switching point both D and S are equal to the observed Q. In the present example this means that S_{t-1} would also be assumed to be equal to Q_{t-1}. Point $t-1$ would still be counted as being only a demand point, however. As a practical matter, it was felt that it was better to make this assumption than to ignore serial correlation problems altogether.

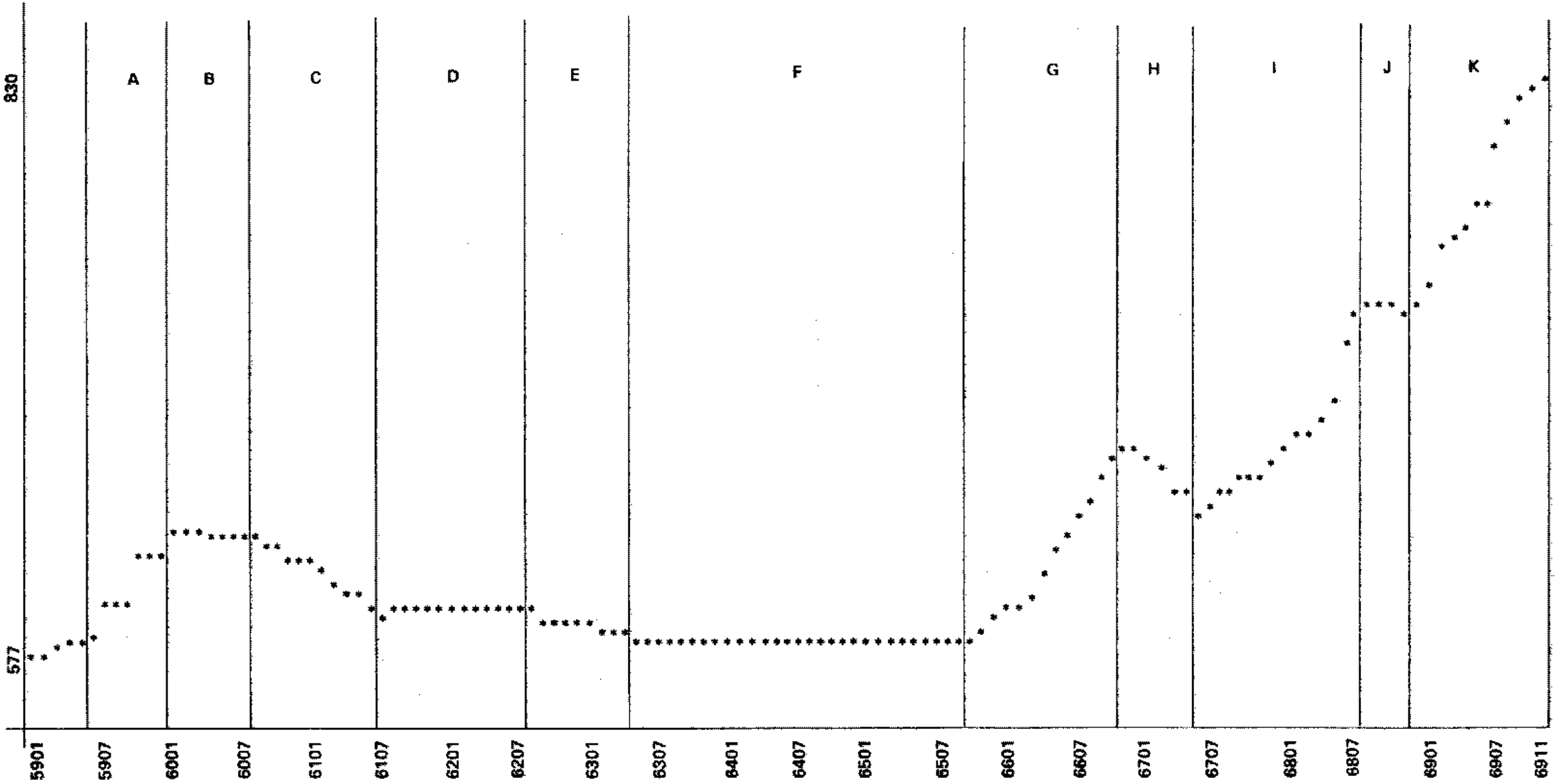

FIGURE 2.—Mortgage rate, *RM*, January, 1959–November, 1969.

The model was next estimated by directional method I, the results of which are presented in lines (D-2) and (S-2) of Table I.[24] The sample periods that were used for these estimates can be obtained from examining the plot of the mortgage rate RM in Figure 2. From June, 1959 on, the demand sample period was taken to include all those months in which the change in RM was either zero or negative, and the supply sample period was taken to include all those months in which the change in RM was either zero or positive. (The A, B, C, ... regions in Figure 2 pertain to directional method II and should be ignored for now.) The results in lines (D-2) and (S-2) are good. The fits are better than the corresponding fits in lines (D-1) and (S-1), and the coefficient estimates are all of the expected signs. One of the more pronounced changes occurred in the demand equation, where the estimate of the coefficient of RM_{t-2} changed from $-.139$ in line (D-1) to $-.241$ in line (D-2).

The model was next estimated by directional method II. The results are presented in lines (D-3) and (S-3) of Table I. This method was implemented as follows. The periods in which RM was unambiguously rising—periods A, G, I, and K in Figure 2—were chosen as definite supply periods, and the periods in which RM was unambiguously falling—periods C, E, and H in Figure 2—were chosen as definite demand periods. The remaining "flat" periods—B, D, F, and J—were left unspecified except for the assumption that each period was either all demand or all supply as opposed to combinations of the two. This produced 16 sets of sample separations to be analyzed, and for each separation the value of $\log L$ in (9) was computed. The largest value of $\log L$ occurred for the pair which included B and F as demand periods and D and J as supply periods. This final pair of sample periods was then "lagged" (see Footnote 16 and related discussion) one, two, and three months to see if any of these three pairs resulted in a larger value of $\log L$ than that produced by the "current" pair. This was not the case, so the current pair was chosen as the best pair.

The results obtained from using directional method II turned out not to be very good. All of the signs in lines (D-3) and (S-3) of Table I are right and the fits are good, but the size of the coefficient estimate of RM_{t-2} in the demand equation appeared to be too large in absolute value. In particular, when the demand equation was extrapolated beyond April, 1967 (the most recent point of the demand sample period), the predicted values were much smaller (by nearly a factor of two in the last half of 1969) than the actual values. This was, of course, in a period when demand, if anything, should have been greater than the (observed) supply. What happened was that by failing to use the information that demand was at least equal to the observed quantity during 1968 and 1969 when the mortgage rate was rising rapidly (and indeed rising to values never before observed in the sample period), directional method II overestimated the effect of the mortgage rate on demand. The failure of directional method II in the present example is thus

[24] The same assumption described in the previous footnote had to be made for this method as well. For example, June, 1967 was a switching point from a demand period into a supply period, and thus for May, 1967 supply was assumed to be equal to demand. May, 1967 was still counted as a demand point, however. This same assumption was made for the application of directional method II as well.

somewhat unique, the poor results being due at least in large part to the extreme behavior of the mortgage rate in the last two years of the sample period. Nevertheless, directional method II does have the disadvantage that by not being able to include sample points in both regimes it utilizes less of the sample period information than does directional method I.

Finally, the quantitative method was used to estimate the model. The results are presented in lines (D-4)–(D-6) and (S-4)–(S-6) of Table I. For the results in lines (D-4) and (S-4) the constraint that the coefficient of $\diagup \Delta RM_t \diagup$ in the demand equation should be the same as the coefficient of $\diagdown \Delta RM_t \diagdown$ in the supply equation was not imposed, and no account was taken of possible simultaneous equation bias from the endogeneity of $\diagup \Delta RM_t \diagup$ and $\diagdown \Delta RM_t \diagdown$.[25] For the results in lines (D-5) and (S-5) the constraint was imposed, but no account was taken of possible simultaneous equation bias.[26] For the results in lines (D-6) and (S-6), account was taken of possible simultaneous bias by using a two-stage least squares technique, but the constraint was not imposed.[27]

Looking at the constrained versus unconstrained results first, the coefficient estimates are remarkably similar. Without the constraint imposed the coefficient estimates of $\diagup \Delta RM_t \diagup$ and $\diagdown \Delta RM_t \diagdown$ are $-.404$ and $-.448$ respectively, and so imposing the constraint resulted in little change in any of the estimates in either equation. (The constrained estimate is $-.408$.) All of the coefficient estimates are of the expected sign in the equations, and in particular the constrained coefficient estimate of $\diagup \Delta RM_t \diagup$ and $\diagdown \Delta RM_t \diagdown$ is negative and significant. The results do confirm the hypothesis that demand is greater than supply when ΔRM_t is positive and less than supply when ΔRM_t is negative.

Looking next at the results in lines (D-6) and (S-6) of Table I, the use of the two-stage least squares technique resulted in smaller estimates for the coefficients of $\diagup \Delta RM_t \diagup$ and $\diagdown \Delta RM_t \diagdown$. This is as expected, since without correcting for simultaneous equation bias the estimates are expected to be biased toward zero. The other coefficient estimates in lines (D-6) and (S-6) were not changed much from their values in lines (D-4) and (S-4). The results thus indicate that there is some degree of simultaneous equation bias in the estimates in lines (D-4) and (S-4) or in lines (D-5) and (S-5) that needs to be corrected. As mentioned in Section II, however, it is not clear for small sample purposes whether consistency should be attained at a cost of some efficiency or whether efficiency should be gained at a cost of consistency. For the present example the results in lines (D-5) and (D-6)

[25] The equations were estimated by the Cochrane-Orcutt iterative technique.

[26] The equations were estimated by a technique that is developed in Fair [2] for estimating a set of equations with serially correlated errors and restrictions across equations.

[27] Because of serial correlation of the error terms, the standard two-stage least squares technique could not be used. The technique that was used is described in Fair [3]. Care must be taken when using this technique to insure that the necessary instruments are included in the first stage regression. In the present case the instruments used were (aside from the dummy variables and the current and lagged value of the work day variable) the constant, t, the housing stock variable lagged once, the housing stock variable lagged twice, RM_{t-1}, RM_{t-2}, RM_{t-3}, $DF6_{t-1}$, $DF6_{t-2}$, $DHF3_{t-2}$, and $DHF3_{t-3}$. For the demand equation, in the first stage regression ΔRM_t was regressed over that part of the sample period for which ΔRM_t was non-negative, and for the supply equation, ΔRM_t was regressed over that part of the sample period for which ΔRM_t was non-positive.

and in lines (S-5) and (S-6) are not sufficiently different to warrant any conclusion as to which estimation technique is better.

In summary, it should be noted that directional method I and the quantitative method, the two methods that gave good results, yielded coefficient estimates that were not much different from the estimates obtained under the assumption of full equilibrium. The significance of the estimates of the coefficient of the $\nearrow \Delta RM_t \nearrow$ and $\searrow \Delta RM_t \searrow$ variables in Table I, however, indicates that the housing market is not always in equilibrium, and that somewhat better results can be obtained by treating the housing market as a disequilibrium market. It is, of course, difficult to know to what extent the results obtained in this section are due to the specific nature of the housing and mortgage market, and it would be desirable to test other markets, or other specifications of the housing and mortgage market, using the methods developed in this paper.

Princeton University

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