Chapter Two

Banks

2.1 THE BASIC EQUATIONS

In Table 2-1 the important symbols used in this chapter are listed in alphabetic order. The first half of the table presents the notation for the non-condensed model, and the second half presents the notation for the condensed model. The notation for the condensed model pertains only to the discussion in Section 2.6.

Each bank, say bank i, receives money from households in the form of savings deposits (SDB_{it}), on which interest is paid, and from firms, households, and the bond dealer in the form of demand deposits ($DDBit$), on which no interest is paid. Each bank lends money to firms and households (LB_{it}) and buys government bills ($VBILLB_{it}$) and bonds ($BONDB_{it}$). Bank loans are one-period loans, bills are one-period securities, and bonds are consols. Each bank holds reserves in the form of deposits with the government (BR_{it}). Each bank sets its own loan rate (RB_{it}). The three main decision variables of each bank are its loan rate, the value of bills and bonds to purchase (VBB_{it}), and the maximum amount of money that it will lend in the period ($LBMAX_{it}$). Banks are assumed not to compete for savings deposits, and the rate paid on all savings deposits is assumed to be the bill rate (r_t).

Table 2-1. Notation for Banks in Alphabetic Order

Non-Condensed Model

Subscript i denotes variable for bank i. Subscript j denotes variable for bank j. Subscript t denotes variable for period t. An e superscript in the text denotes an expected value of the variable.

$BONDB_{it}$	= number of bonds held, each bond yielding one dollar per period
BR_{it}	= actual reserves
BR^*_{it}	= required reserves
d_1	= profit tax rate
d_2	= penalty tax rate on the composition of banks' portfolios
DDB_{it}	= demand deposits
$DIVB_{it}$	= dividends paid
$EMAXDD_i$	= largest error the bank expects to make in overestimating its demand deposits for any period
$EMAXSD_i$	= largest error the bank expects to make in overestimating its savings deposits for any period
$FUNDS^e_{it}$	= amount that the bank knows it will have available to lend to households and firms and to buy bills and bonds even if it overestimates its demand and savings deposits by the maximum amounts
g_1	= reserve requirement ratio
g_2	= no-tax proportion of banks' portfolios held in bills and bonds
L_t	= total value of loans of the bank sector
LB_{it}	= value of loans
$LBMAX_{it}$	= maximum value of loans that the bank will make
LUN_t	= total unconstrained demand for loans
r_t	= bill rate
R_t	= bond rate
RB_{it}	= loan rate (of bank i)
RB_{jt}	= loan rate (of bank j)
$\overline{RB}_t$	= average loan rate in the economy
SDB_{it}	= savings deposits
$TAXB_{it}$	= taxes paid
$T+1$	= length of decision horizon
VBB_{it}	= value of bills and bonds that the bank chooses to purchase $[VBILLB_{it} + BONDB_{it}/R_t]$
$VBILLB_{it}$	= value of bills held
ΠB_{it}	= before-tax profits

Condensed Model (For equations in Table 2-4 only.)

Subscript t denotes variable for period t. Only notation that differs from the notation for the non-condensed model is presented here.

$EMAXDD$	= largest error the bank sector expects to make in overestimating its demand deposits for any period
$EMAXSD$	= largest error the bank sector expects to make in overestimating its savings deposits for any period
$FUNDS^e_t$	= amount the bank sector knows it will have available to lend to households and firms and to buy bills and bonds even if it overestimates its demand and savings deposits by the maximum amounts
$LBMAX_t$	= maximum value of loans that the bank sector will make
RL_t	= loan rate of the bank sector
SD_t	= savings deposits of the bank sector
$TAXB_t$	= taxes paid by the bank sector
VBB_t	= value of bills and bonds that the bank sector chooses to purchase $[VBILLB_t + BONDB_t/R_t]$
$VBILLB_t$	= value of bills held by the bank sector
ΠB_t	= before-tax profits of the bank sector

The basic equations for bank i for period t are the following:

$$VBB_{it} = VBILLB_{it} + BONDB_{it}/R_t, \text{ [value of bills and bonds held]} \tag{2.1}$$

$$\Pi B_{it} = RB_{it}LB_{it} + r_t VBILLB_{it} + BONDB_{it} - r_t SDB_{it} + (BONDB_{it}/R_{t+1} - BONDB_{it}/R_t), \text{ [before-tax profits]} \tag{2.2}$$

$$TAXB_{it} = d_1\ \Pi B_{it} + d_2\ [VBB_{it} - g_2(VBB_{it} + LB_{it})]^2, \text{ [taxes paid]} \tag{2.3}$$

$$DIVB_{it} = \Pi B_{it} - TAXB_{it}, \text{ [dividends paid]} \tag{2.4}$$

$$\begin{aligned} BR_{it} &= DDB_{it} + SDB_{it} - LB_{it} - VBB_{it} - (BONDB_{it}/R_{t+1} - BONDB_{it}/R_t) \\ &= DDB_{it} + SDB_{it} - LB_{it} - VBILLB_{it} - BONDB_{it}/R_{t+1}, \text{ [actual reserves]} \end{aligned} \tag{2.5}$$

$$BR^*_{it} = g_1 DDB_{it}, \text{ [required reserves]} \tag{2.6}$$

$$BR_{it} \geqslant BR^*_{it}. \text{ [actual reserves must be greater than or equal to required reserves]} \tag{2.7}$$

Equation (2.1) merely defines the value of bills and bonds held. Since bonds are consols and since each bond is assumed to yield one dollar each period, the value of bonds held is merely the number held divided by the bond rate, $BONDB_{it}/R_t$. Equation (2.2) defines before-tax profits. The first three terms on the right-hand side of the equation are the interest revenue received on loans, bills, and bonds, respectively.[a] The fourth term is the interest paid on savings deposits. The last term is the capital gain or loss made on bonds held in period t.

Taxes are defined in Equation (2.3), where d_1 is the profit tax rate. With respect to the second term on the right-hand-side of the equation, the government is assumed through its taxing policy to try to induce banks to hold a certain proportion, g_2, of their assets in bills and bonds. In practice, commercial banks and other financial intermediaries are under certain pressures to hold particular kinds of securities, and here these pressures are assumed to take the form of government taxing policy. If, in the model, banks were not induced in some way to hold bills and bonds, they would never want to hold bills and bonds as long as their loan rates were higher than the bill rate. The introduction of government taxing policy is a simple way of explaining why banks hold more than one kind of asset.

In Equation (2.3), bank i is assumed to be taxed at rate d_2 on the square of the difference between the value of bills and bonds held and g_2 times the value of loans issued plus bills and bonds held. Since capital gains and losses are included in the definition of profits, Equation (2.3) also reflects the assumption that capital gains and losses are taxed as regular income. Bank i is assumed not to retain any earnings, so that the level of dividends, as defined in Equation (2.4), is merely the difference between before-tax profits and taxes.

Bank reserves are defined in Equation (2.5). Since bank i pays out in the form of taxes and dividends any capital gains made in the period (and conversely for capital losses), and yet does not receive any actual cash flow from the capital gains, capital gains take away from (and conversely capital losses add to) bank reserves, as specified in (2.5). Required reserves are defined in Equation (2.6). For simplicity, no reserve requirements are placed on savings deposits. Actual reserves must be greater than or equal to required reserves, as indicated in (2.7).

2.2 THE FORMATION OF EXPECTATIONS

Let $T+1$ be the length of the decision horizon. In order for the bank to solve its control problem at the beginning of period t, it must form expectations of a number of variables for periods t through $t+T$. Bank i is assumed to form the following expectations.[b]

$$\frac{RB^e_{jt}}{RB_{jt-1}} = \left(\frac{RB_{it-1}}{RB_{jt-1}}\right)^{\alpha_1} \left(\frac{r_t}{r_{t-1}}\right)^{\alpha_2}, \alpha_1 > 0, \alpha_2 > 0, \text{ [expected loan rate of bank j for period t]} \quad (2.8)$$

$$\frac{RB^e_{jt+k}}{RB^e_{jt+k-1}} = \left(\frac{RB_{it+k-1}}{RB^e_{jt+k-1}}\right)^{\alpha_1}, \text{ [expected loan rate of bank } j \text{ for period } t+k \; (k = 1, 2, \ldots, T)] \quad (2.9)$$

$$\overline{RB}^e_{t+k} = (RB_{it+k} \cdot RB^e_{jt+k})^{\frac{1}{2}}, \text{ [expected average loan rate for period } t+k \; (k = 0, 1, \ldots, T)] \quad (2.10)$$

$$LUN^e_t = LUN_{t-1}\left(\frac{\overline{RB}_{t-1}}{\overline{RB}^e_t}\right)^{\alpha_3}, \alpha_3 > 0, \text{ [expected aggregate unconstrained demand for loans for period } t] \quad (2.11)$$

$$LUN^e_{t+k} = LUN^e_{t+k-1}\left(\frac{\overline{RB}^e_{t+k-1}}{\overline{RB}^e_{t+k}}\right)^{\alpha_3}, \text{ [expected aggregate unconstrained demand for loans for period } t+k \; (k = 1, 2, \ldots, T)] \quad (2.12)$$

$$L^e_{t+k} = LUN^e_{t+k}, \text{ [expected aggregate constrained demand for loans for period } t+k \; (k=0, 1, \ldots, T)] \quad (2.13)$$

$$\frac{LB^e_{it}}{L^e_t} = \frac{LB_{it-1}}{L_{t-1}}\left(\frac{RB_{it}}{RB^e_{jt}}\right)^{\alpha_4}, \alpha_4 < 0, \text{ [expected market share of loans for period } t] \quad (2.14)$$

$$\frac{LB^e_{it+k}}{L^e_{t+k}} = \frac{LB^e_{it+k-1}}{L^e_{t+k-1}} \left(\frac{RB_{it+k}}{RB^e_{jt+k}}\right)^{\alpha_4}, \text{ [expected market share of loans for period } t+k\ (k=1,2,\ldots,T)] \quad (2.15)$$

$$DDB^e_{it+k} = DDB_{it-1}, \text{ [expected level of demand deposits for period } t+k\ (k=0,1,\ldots,T)] \quad (2.16)$$

$$SDB^e_{it+k} = SDB_{it-1}, \text{ [expected level of savings deposits for period } t+k\ (k=0,1,\ldots,T)] \quad (2.17)$$

$$r^e_{t+k} = r_t, \text{ [expected bill rate for period } t+k\ (k=1,2,\ldots)] \quad (2.18)$$

$$\frac{1}{R^e_{t+k}} = \frac{1}{(1+r^e_{t+k})} + \frac{1}{(1+r^e_{t+k})(1+r^e_{t+k+1})} + \frac{1}{(1+r^e_{t+k})(1+r^e_{t+k+1})(1+r^e_{t+k+2})} + \ldots$$

$$= \frac{1}{r_t}. \text{ [equation determining expected bond rate for period } t+k\ (k=0,1,\ldots,T+1)] \quad (2.19)$$

The first term on the right-hand side of Equation (2.8) reflects the fact that bank i expects its rate setting behavior in period $t-1$ to have an effect on bank j's rate setting behavior in period t. The second term is designed to represent the effect of general market conditions on bank i's expectation of bank j's rate. The bond dealer sets the bill rate for period t at the end of period $t-1$, and bank i knows the bill rate for period t at the time that it makes its decisions for period t. If the bill rate for period t has changed, then bank i is assumed to expect that this change will have an effect in the same direction on the rate that bank j sets in period t.

Bank i must also form expectations of bank j's rate for periods $t+1$ and beyond. These expectations are specified in Equation (2.9), which is the same as Equation (2.8) without the final term. Equation (2.9) means that bank i expects that bank j is always adjusting its rate toward bank i's rate. If bank i's rate is constant over time, then bank i expects that bank j's rate will gradually approach this value.

In Equation (2.10) bank i's expectation of the average loan rate is taken to be the geometric average of its rate and its expectation of bank j's rate. Without loss of generality, there is assumed to be only one other bank, bank j, in existence. It should be obvious how the number of other banks in existence can be generalized to be more than one. There is nothing inconsistent in the model with there being a relatively large number of other banks in existence. The geometric average is used in (2.10) rather than the arithmetic average to make the solution of the model easier. Bank i expects that the aggregate unconstrained demand for loans is a function of the average loan rate, as specified in Equations (2.11) and (2.12).[c] The aggregate unconstrained demand for loans in, say, period

$t-1$ (LUN_{t-1}) is what would have been the demand for loans on the part of firms and households had they not been subject to any constraints. Each bank is assumed to be aware of this demand. The aggregate constrained demand for loans (L_{t-1}) is the actual value of loans made in period $t-1$. Equation (2.13) states that bank i expects that firms and households will not be constrained in their borrowing behavior in periods t and beyond. The expected aggregate constrained demand for loans is assumed in Equation (2.13) to be equal to the expected aggregate unconstrained demand for loans for each period. As will be seen below, bank i does not itself expect to turn any customers away, and so Equation (2.13) merely states that bank i also does not expect any customers in the aggregate to be turned away.

Equation (2.14) determines bank i's expectation of its market share for period t and reflects the assumption that a bank expects that its market share is a function of its rate relative to the rates of other banks. The equation states that bank i's expected market share for period t is equal to last period's market share times a function of the ratio of bank i's rate for period t to the expected rate of bank j for period t. Equation (2.15) is a similar equation for periods $t+1$ through $t+T$.

It should be noted that the market share for period $t-1$ on the right-hand side of Equation (2.14) is the ratio of the actual value of loans of bank i in period $t-1$ to the *actual* value of aggregate loans in period $t-1$ (LB_{it-1}/L_{t-1}) and is not the ratio of the actual value of bank i's loans to the aggregate *unconstrained* demand for loans (LB_{it-1}/LUN_{t-1}). Since bank i is assumed to know both L_{t-1} and LUN_{t-1}, the latter specification is a possibility.

The justification for the use of LB_{it-1}/L_{t-1} is as follows. L_t^e is bank i's expectation of the aggregate unconstrained (and constrained) demand for loans for period t. Of the potential customers represented by this amount, some will come to bank i during the period. How many come depends on how large a part bank i is of the market in period $t-1$ and on the relative loan rates. Now, a good measure of how large a part bank i is of the market in period $t-1$ is its actual market share in period $t-1$. This measure is a better measure than LB_{it-1}/LUN_{t-1}, since the latter does not represent in any direct sense bank i's participation in the market. If LUN_{t-1} is greater than L_{t-1}, only a part of the unsatisfied customers represented by this amount are likely to have been turned away by bank i. The rest of the customers would not have sampled bank i in the period. Therefore, it seems more in the spirit of the search literature to use the actual market share on the right-hand side of Equation (2.14).

As should be evident from the discussion in the next section, Equations (2.8)–(2.15) are quite important in determining the rate setting behavior of bank i. Two similar sets of equations are also postulated in Chapter Three regarding the price setting and wage setting behavior of a firm. The two most important assumptions underlying Equations (2.8)–(2.15) are that bank i expects that its rate setting behavior has an effect on bank j's rate setting behavior and that bank i expects that its market share is a function of its rate relative to bank j's rate. The equations can be easily modified if there is more than just one other bank in existence. Equations (2.8) and (2.9) would hold for each bank. Equation (2.10) would be the geometric average over all banks. Equations (2.11)–(2.13) would remain the same, and Equations (2.14) and (2.15) would be changed either to include all the ratios of bank i's rate to the other banks' rates, or to include the ratio of bank i's rate to the average of the other banks' rates.

In Equations (2.16)–(2.18) bank i is assumed to expect that the values of demand deposits, savings deposits, and the bill rate for all future periods will be the same as the last observed values of these variables. Equation (2.19) determines the expected bond rate. The right-hand side of the equation is the present discounted value of a perpetual stream of one-dollar payments, the discount rates being the expected future bill rates. The right-hand side of the equation can thus be considered to be the expected price of a bond for period $t+k$, and so the reciprocal of this expression can be considered to be the expected bond rate for period $t+k$. This assumption, of course, ignores the fact that the expected value of a ratio is not equal to the ratio of the expected values, but this type of problem is ignored all the way through this study by the converting of stochastic control problems into deterministic control problems in the manner discussed in Section 1.3.

The assumptions in (2.16) and (2.17), that bank i expects no change in its demand and savings deposits from the last observed values, are important and typical of many expectational assumptions made in the model. Whenever an expectational assumption had to be made that was either not concerned with market share situations or for which no obvious assumption was available, the simple assumption of no change from the last observed value was made. The aim was not to complicate the model any more than seemed necessary to capture important expectational issues.

As long as lagged values have some effect on expectations of current and future values, assumptions like (2.16) and (2.17) should not be too unrealistic. It should also be noted that because of the assumption in (2.18), that bank i expects no change in the future bill rates from the last observed rate, the expected bond rates in (2.19) are simply equal to the current bill rate. It was mentioned in Section 1.2 that the only reason bonds were included in the model at all was to account for the effects of capital gains and losses, and so nothing is really lost in the model by having the bill rate and bond rate always be equal.

2.3 BEHAVIORAL ASSUMPTIONS

The objective of a bank is to maximize the present discounted value of expected future after-tax profits. The discount rate is assumed to be the bill rate. The objective function of bank i at the beginning of period t is:

$$OBJB_{it} = \frac{\Pi B^e_{it} - TAXB^e_{it}}{(1+r_t)} + \frac{\Pi B^e_{it+1} - TAXB^e_{it+1}}{(1+r_t)(1+r^e_{t+1})} + \dots + \frac{\Pi B^e_{it+T} - TAXB^e_{it+T}}{(1+r_t)(1+r^e_{t+1})\dots(1+r^e_{t+T})}, \quad (2.20)$$

where $\Pi B^e_{it+k} - TAXB^e_{it+k}$ is the expected value of after-tax profits for period $t+k$ $(k=0,1,\dots,T)$.

Three of the decision variables of bank i are its loan rate, RB_{it+k}, the value of bills to purchase, $VBILLB_{it+k}$, and the number of bonds to purchase, $BONDB_{it+k}$ $(k=0,1,\dots,T)$. Given paths of these three variables, the corresponding value of the objective function can be computed as follows.

1. Given bank i's rate path, bank i's expectation of bank j's rate path can be computed from (2.8) and (2.9). The path of the expected average loan rate can then be computed from (2.10), followed by the path of the expected aggregate unconstrained demand for loans from (2.11) and (2.12). The path of the expected aggregate constrained demand for loans can then be computed from (2.13), followed by bank i's expectation of the demand for its own loans from (2.14) and (2.15).
2. The paths of expected demand deposits, savings deposits, the bill rate, and the bond rate are determined in Equations (2.16)–(2.19). Given these four paths and given the paths discussed in 1, the paths of expected profits and taxes can be computed from (2.2) and (2.3),[d] which then means that the value of the objective function can be computed.

A few general remarks can now be made regarding the control problem of a bank. A bank expects that it will gain customers by lowering its rate relative to the expected rates of other banks. The main expected cost to a bank from doing this, in addition to the lower price it is charging per loan, is that it will have to pay more and more taxes the further it deviates from holding proportion g_2 of its portfolio in bills and bonds. It is also the case that a bank expects that other banks will follow it if it lowers its rate, so that it does not expect to be able to capture an ever increasing share of the market without further and further rate reductions.

A bank expects that it will lose customers by raising its rate relative to the expected rates of other banks. Again, the main cost from doing this, in addition to the lost customers, is the higher taxes that must be paid from not holding proportion g_2 of its portfolio in bills and bonds. On the plus side, a bank expects that other banks will follow it if it raises its rate, so that it will not lose an ever increasing share of the market without further and further rate increases.

With respect to a bank's decision regarding bills and bonds, equation (2.19) means that a bank expects that the before-tax, one-period rate of return on bonds, including capital gains and losses, for a given period will be the same as the expected bill rate for that period. Since capital gains and losses are taxed at the same rate as other income, the expected after-tax rates of return on bills and bonds are also the same. Because of this, banks are assumed to be indifferent between holding bills and bonds, and so instead of determining two variables, $VBILLB_{it}$ and $BONDB_{it}$, a bank can be considered, given R_t, as determining only VBB_{it}.

The main constraint facing a bank is the reserve requirement constraint (2.7). A bank expects to receive in funds in period $t+k$, DDB^e_{it+k} + SDB^e_{it+k}, of which $g_1 DDB^e_{it+k}$ is needed to meet the reserve requirement. Therefore, $(1-g_1)DDB^e_{it+k} + SDB^e_{it+k}$ is the expected amount available for period $t+k$ to lend to households and firms and to buy bills and bonds. A bank is assumed, however, to have to prepare for the possibility that it overestimates its demand and savings deposits. A bank is assumed from past experience to have a good idea of the largest error it is likely to make in overestimating its demand and savings deposits. Call the error for demand deposits $EMAXDD_i$ and the error for savings deposits $EMAXSD_i$. For simplicity, these expected maximum errors are assumed not to change over time. The quantity $(1-g_1)$ $(DDB^e_{it+k} - EMAXDD_i) + (SDB^e_{it+k} - EMAXSD_i)$ is the amount that bank i knows it will have available in period $t+k$ to lend to households and firms and to buy bills and bonds even if it overestimates its demand and savings deposits by the maximum amounts. Denote this quantity as $FUNDS^e_{it+k}$:

$$FUNDS^e_{it+k} = (1-g_1)(DDB^e_{it+k} - EMAXDD_i) + (SDB^e_{it+k} - EMAXSD_i). \quad (2.21)$$

Now, given a path of bank i's loan rate, it was seen from 1 above that bank i can compute the path of its expected loans (LB^e_{it+k}, $k=0,1,\ldots,T$). In order to make sure of meeting the reserve requirement constraint, bank i is assumed to behave by choosing the path of its loan rate and the path of the value of bills and bonds to buy (VBB_{it+k}, $k = 0,1,\ldots,T$) so as to satisfy the constraint that

$$LB^e_{it+k} + VBB_{it+k} = FUNDS^e_{it+k}, \; k=0,1,\ldots,T. \quad (2.22)$$

By satisfying equation (2.22), bank i is assured that it will have enough funds to meet the expected loan demand each period, given its path of the value of bills and bonds to buy. Once a bank decides at the beginning of period t the value of bills and bonds to purchase in the period, it is assumed that the bank must purchase this value.

This is still not the end of the story, however, for bank i must also prepare for the possibility that it underestimates the demand for its loans at the loan rate path that it has chosen. Bank i is assumed to prepare for this possibility by announcing to households and firms the maximum amount of money that it will lend in each period, in addition to announcing the loan rate. The maximum amount each period ($LBMAX_{it+k}$, $k = 0,1, \ldots ,T$) is assumed to be equal to the expected loan demand for that period:

$$LBMAX_{it+k} = LB^{e}_{it+k},\ k=0,1,\ldots,T. \tag{2.23}$$

Bank i is now assured of meeting its reserve requirement. It will always have at least amount $FUNDS^{e}_{it+k}$ at its disposal, and it will never use more than this amount to lend to households and firms and to buy bills and bonds. The procedure just described means, of course, that a bank expects to hold some amount of excess reserves most of the time. Only in the extreme case where it overestimates its demand and savings deposits by the full amounts $EMAXDD_i$ and $EMAXSD_i$, and also lends to households and firms the maximum amount of money that it set, will it end up with zero excess reserves.

Although in practice commercial banks and some other kinds of financial intermediaries can usually meet unexpected situations by borrowing from a monetary authority, the procedure just described by which banks account for unexpected situations in the model is not necessarily unrealistic. Commercial banks and other financial intermediaries are under basic constraints of the kind considered above, and it is not unreasonable to assume that these constraints play an important role in their decision making processes. Also, if a bank can hold negative excess reserves in the short run by borrowing from a monetary authority, all this really means in the present context is that the bank would maximize (2.20) subject to the constraint that $LB^{e}_{it+k} + VBB_{it+k}$ in (2.22) be equal to $FUNDS^{e}_{it+k}$ *plus* some positive number. The positive number might be, for example, the maximum that the bank could expect to borrow from the monetary authority in an emergency situation.

It is likewise not necessarily unrealistic to assume that banks must buy in the period the value of bills and bonds that they chose to buy at the beginning of the period. Although in practice one bank can sell bills and bonds to another bank to get more funds to lend to households and firms, in the aggregate this cannot be done. In the aggregate the government determines the number of bills and bonds to have outstanding, and the private sector must behave within this constraint. In the model the bond dealer absorbs each period the difference between the supply of bills and bonds from the government and the demand from the banks, so the assumption that banks cannot change their decisions on the value of bills and bonds to buy during the period merely simplifies the specification of the way that transaction takes place during the period. Any discrepancy between the supply from the government and the demand from the banks in the current period affects the bill and bonds rates set by the bond dealer for the next period.

2.4 THE SOLUTION OF THE CONTROL PROBLEM

It was seen in the last section that given the paths of the loan rate and the value of bills and bonds to buy, the corresponding value of the objective function can be computed. In order to solve the control problem of bank i, an algorithm was written to search over various loan rate paths. The base path, from which other paths were tried, was taken to be the path in which the proportion of bills and bonds held each period was equal to the no-penalty-tax proportion g_2. The loan rate path corresponding to this situation is computed as follows.

First, VBB_{it+k} is set equal to $g_2 FUNDS^e_{it+k}$, and LB^e_{it+k} is set equal to $FUNDS^e_{it+k} - VBB_{it+k}$ $(k=0,1,\ldots,T)$. Now, for period t, given the values for period $t-1$, Equations (2.8), (2.10), (2.11), (2.13), and (2.14) form a system of five equations in six unknowns: RB^e_{jt}, $\overline{RB}^e_t$, LUN^e_t, L^e_t, LB^e_{it}, and RB_{it}. Given a value for LB^e_{it}, the system reduces to a system of five equations in five unknowns, which can be solved recursively to obtain a value for RB_{it}. For period $t+1$, given the values for period t, Equations (2.9), (2.10), (2.12), (2.13), and (2.15) likewise form a system of five equations in six unknowns. Given a value for LB^e_{it+1}, a value for RB_{it+1} can be obtained. This process can then be repeated for periods $t+2,\ldots,t+T$ to obtain the base loan rate path.

Given the base loan rate path, it is straightforward to search over alternative paths. Given a value of RB_{it} and given values for period $t-1$, equations (2.8), (2.10), (2.11), (2.13), and (2.14) can be solved for RB^e_{jt}, $\overline{RB}^e_t$, LUN^e_t, L^e_t, and LB^e_{it}. Once LB^e_{it} has been determined in this way, the value of VBB_{it} is merely the difference between $FUNDS^e_{it}$ and LB^e_{it}. Values for periods $t+1$ and beyond can be obtained in the same way by solving Equations (2.9), (2.10), (2.12), (2.13), and (2.15). The algorithm was programmed to search in one direction until the value of the objective function decreased and then to try other directions. Particular importance was attached to searching over values of RB_{it}, since this is the value actually used in the solution of the overall model.

2.5 SOME EXAMPLES OF SOLVING THE CONTROL PROBLEM OF BANK *i*

PARAMETER VALUES AND INITIAL CONDITIONS

The parameter values and initial conditions that were used for the first example are presented in Table 2-2. The most important parameters are d_2, the penalty tax rate on portfolio composition, α_1, the measure of the extent to which bank i

Table 2-2. Parameter Values and Initial Conditions for the Control Problem of Bank *i*

Parameter	*Value*	
$T+1$	30	
g_1	0.1667	
g_2	0.2956	
d_1	0.5	
d_2	0.0028	
α_1	0.5	
α_2	0.4	
α_3	0.2	
α_4	−3.6	
$EMAXDD_i$	1.9	
$EMAXSD_i$	10.1	
Variable	*Value*	
DDB_{it-1}	96.1	
SDB_{it-1}	506.7	
LB_{it-1}	405.1	
L_{t-1}	810.2	
LUN_{t-1}	810.2	
RB_{it-1}	0.0750	
RB_{jt-1}	0.0750	
r_t	0.0650	
r_{t-1}	0.0650	
$FUNDS^e_{it}$	575.1	$= (1-g_1)(DDB_{it-1}-EMAXDD_i)+(SDB_{it-1}-EMAXSD_i)$
VBB_{it}	170.0	$= FUNDS^e_{it} - LB_{it-1}$
$\dfrac{VBB_{it-1}}{LB_{it-1}+VBB_{it-1}}$	0.2956 $= g_2$	

expects bank j to respond to bank i's rate setting behavior, α_2, the measure of the extent to which bank i expects bank j to change its rate for period t as a result of a change in the bill rate, and α_4, the measure of the extent to which bank i loses or gains market share as its rate deviates from bank j's rate. The market share parameter, α_4, is more important than the parameter α_3, which is the measure of the extent to which bank i expects the aggregate demand for loans to change as a function of the average loan rate in the economy. More will be said about this in Chapter Six.

The parameter values and initial conditions were chosen, after some experimentation, so that the optimum values of each control variable for periods t through $t+T$ would be essentially the same as the initial value for period $t-1$. This was done to make it easier to analyze the effects on the behavior of the bank of changing various initial conditions. As can be seen from Table 2-2, the initial conditions for the first example correspond to bank i's having half of the loans in period $t-1$. The loan rates of bank i and bank j in period $t-1$ are the same. The bill rate is one percentage point lower than the loan rates. The ratio of bills and bonds to loans plus bills and bonds in period $t-1$ is equal to the no-tax ratio g_2. The aggregate unconstrained demand for loans in period $t-1$ is the same as the constrained demand. The length of the decision horizon is 30 periods.

THE RESULTS

The results of solving the control problem of bank i for the parameter values and initial conditions in Table 2-2 are presented in the first row of Table 2-3. Only a small subset of the results is presented in Table 2-3, as it is not feasible to present all 30 values for each variable. Values for the first two periods are presented for bank i's loan rate, its expectation of bank j's loan rate, its expectation of the aggregate demand for loans, its expectation of the demand for its own loans, its expectation of its market share, the value of bills and bonds to purchase, and its expectation of the ratio of the value of bills and bonds held to the value of loans plus bills and bonds held. The values in the first row of Table 2-3 for each variable are equal to the corresponding initial value in Table 2-2, which reflects the way the parameter values and initial conditions were chosen.

One important reaction of a bank is how the bank responds to a change in its demand or savings deposits. For the results in row 2 in Table 2-3, $FUNDS_{it}^{e}$ was increased by 5.0 percent. An increase in $FUNDS_{it}^{e}$ can come about by an increase in period $t-1$ of either demand deposits or savings deposits or by a decrease in the reserve requirement ratio. Because of the expectational assumptions regarding demand and savings deposits, a 5.0 percent increase in $FUNDS_{it}^{e}$ means that bank i expects all future values of this variable to be 5.0 percent higher as well.

From the results in row 2 it can be seen that this change caused bank i to lower its loan rate for periods t and beyond in an attempt to increase the demand for its loans. Since bank i expected that bank j's rate would not respond to this change in bank i's rate until period $t+1$, bank i expected to increase its share of the market from 0.5000 to 0.5241 in period t. The aggregate demand for loans was expected to increase slightly in period t from 810.2 to 811.3 because of the lower average loan rate caused by bank i lowering its rate. Bank i also chose to raise its ratio of bills and bonds to loans plus bills and bonds from 0.2956 to 0.2960. This slight substitution into bills and bonds from the no-tax amount was caused in effect by the lower loan rate relative to the bill rate.

The values of all of the variables for period $t+1$ were essentially the same as the values for period t except for the value of the loan rate. Bank i found it to its advantage to lower its loan rate by ten basis points for period t and then to raise the rate back by five basis points for period $t+1$. This move enabled bank i to increase its expected market share by enough to absorb the extra loanable funds it expected to have at its disposal for periods t and beyond.

For the results in row 3 in Table 2-3, $FUNDS_{it}^{e}$ was decreased by 5.0 percent. The results in this case were essentially the opposite to those in row 2.

Table 2-3. Results of Solving the Control Problem of Bank *i*

Initial Conditions from Table 2-2 except:	RB_{it}	RB_{it+1}	RB^e_{jt}	RB^e_{jt+1}	L^e_t	L^e_{t+1}	LB^e_{it} ($LBMAX_{it}$)	LB^e_{it+1} ($LBMAX_{it+1}$)
1. No exceptions	0.0750	0.0750	0.0750	0.0750	810.2	810.2	405.1	405.1
2. $FUNDS^e_{it}$ = 603.9 (+5.0%)	0.0740 (−1.3%)	0.0745	0.0750	0.0745	811.3	811.3	425.1	425.3
3. $FUNDS^e_{it}$ = 546.3 (−5.0%)	0.0761 (+1.5%)	0.0755	0.0750	0.0755	809.1	809.1	384.8	384.8
4. a	0.0760 (+1.3%)	0.0755	0.0750	0.0755	849.6	849.6	405.1	405.1
5. b	0.0740 (−1.3%	0.0745	0.0750	0.0745	770.8	770.8	404.9	405.0
6. r_t = 0.0683 (+5.0%)	0.0764 (+1.9%)	0.0764	0.0765	0.0764	807.1	807.1	404.9	405.0
7. r_t = 0.0618 (−5.0%)	0.0736 (−1.9%)	0.0735	0.0735	0.0735	813.4	813.5	405.3	405.2
8. RB_{jt-1} = 0.0788 (+5.0%)	0.0768 (+2.4%)	0.0769	0.0769	0.0769	810.2	810.2	405.3	405.2
9. RB_{jt-1} = 0.0713 (−5.0%)	0.0731 (−2.5%)	0.0731	0.0731	0.0731	810.2	810.2	404.9	405.0
10. g_2 = 0.3104 (+5.0%)	0.0754 (+0.5%)	0.0752	0.0750	0.0752	809.7	809.7	396.6	396.6
11. g_2 = 0.2808 (−5.0%)	0.0746 (−0.5%)	0.0748	0.0750	0.0748	810.7	810.7	413.4	413.5

Table 2-3. (continued)

Initial Conditions from Table 2-2 except:	$\frac{LB^e_{it}}{L^e_t}$	$\frac{LB^e_{it+1}}{L^e_{t+1}}$	VBB_{it}	VBB_{it+1}	$\frac{VBB_{it}}{LB^e_{it}+VBB_{it}}$	$\frac{VBB_{it+1}}{LB^e_{it+1}+VBB_{it+1}}$
1. No exceptions	0.5000	0.5000	170.0	170.0	0.2956	0.2956
2. $FUNDS^e_{it}$ = 603.9 (+5.0%)	0.5241	0.5242	178.7 (+5.1%)	178.6	0.2960	0.2958
3. $FUNDS^e_{it}$ = 546.3 (–5.0%)	0.4757	0.4757	161.5 (–5.0%)	161.5	0.2956	0.2956
4. *a*	0.4768	0.4768	170.0 (+0.0%)	170.0	0.2956	0.2956
5. *b*	0.5253	0.5255	170.2 (+0.1%)	170.1	0.2960	0.2958
6. r_t = 0.0683 (+5.0%)	0.5017	0.5018	170.2 (+0.1%)	170.1	0.2959	0.2958
7. r_t = 0.0618 (–5.0%)	0.4983	0.4981	169.8 (–0.1%)	169.9	0.2952	0.2954
8. RB_{jt-1} = 0.0788 (+5.0%)	0.5002	0.5001	169.8 (–0.1%)	169.9	0.2953	0.2954
9. RB_{jt-1} = 0.0713 (–5.0%)	0.4998	0.4999	170.2 (+0.1%)	170.1	0.2960	0.2958
10. g_2 = 0.3104 (+5.0%)	0.4898	0.4898	178.5 (+5.0%)	178.5	0.3104	0.3104
11. g_2 = 0.2808 (–5.0%)	0.5100	0.5101	161.7 (–4.9%)	161.6	0.2812	0.2810

[a] LB_{it-1} = 425.4 (+5.0%), L_{t-1} = 850.7 (+5.0%), LUN_{t-1} = 850.7 (+5.0%)

[b] LB_{it-1} = 384.8 (–5.0%), L_{t-1} = 769.7 (–5.0%), LUN_{t-1} = 769.7 (–5.0%)

The bank increased its loan rate to lower its expected market share, decreased the value of bills and bonds to purchase, and decreased the maximum amount of money that it will lend to firms and households. In this case, however, the bank did not choose to substitute away from bills and bonds as a result of the higher loan rate relative to the bill rate.

For the results in row 4 in Table 2-3, the value of bank i's loans in period $t-1$ was increased by 5.0 percent, along with a 5.0 percent increase in the aggregate unconstrained and constrained demands for loans. This change caused bank i to increase its loan rate for periods t and beyond. The loan rate was increased to lower the bank's market share to the point where the expected demand for its loans was equal to what the demand was in row 1. This meant that the value of bills and bonds to purchase was not changed. Since $FUNDS_{it}^{e}$ was not changed for this run, the sum of the value of bills and bonds to purchase (VBB_{it}) and bank i's expected loans (LB_{it}^{e}) could not be changed, and so with LB_{it}^{e} remaining unchanged, VBB_{it} must remain unchanged. The results in row 5 in Table 2-3, based on a 5.0 decrease in loans, are essentially the opposite to those in row 4. For the results in row 5, however, the bank chose to substitute into bills and bonds slightly as a result of the lower loan rate relative to the bill rate. The sum of LB_{it}^{e} and VBB_{it}^{e} was still, of course, unchanged, which meant that LB_{it}^{e} was decreased slightly.

For the results in row 6 in Table 2-3, the bill rate for period t was increased by 5.0 percent. This caused bank i to increase its expectation of bank j's rate for period t from 0.0750 to 0.0765. Bank i was led to increase its loan rate one basis point less than this and thus increase its share of the market slightly. The proportion of bills and bonds to loans plus bills and bonds was increased from 0.2956 to 0.2959. Bank i's expectation of the aggregate demand for loans for period t decreased from 810.2 to 807.1 due to the higher loan rates. The results in row 6 thus show that there is some slight substitution into bills and bonds from loans when the bill rate rises. Since $FUNDS_{it}^{e}$ was not changed, the slightly higher value of VBB_{it} implied a slightly lower value of LB_{it}^{e}. The results in row 7 in Table 2-3, based on a 5.0 decrease in the bill rate, are opposite to those in row 6.

For the results in row 8 in Table 2-3, bank j's loan rate for period $t-1$ was increased by 5.0 percent to 0.0788. This caused bank i to increase its expectation of bank j's rate for period t to 0.0769 from the 0.0750 in row 1. Bank i increased its loan rate one basis point less than this and thus increased its share of the market slightly. The proportion of bills and bonds to loans plus bills and bonds was decreased from 0.2956 to 0.2953, which meant that there was some substitution into loans from bills and bonds because of the higher loan rate relative to the bill rate. The results in row 9, based on a 5.0 percent decrease in bank j's rate, are opposite to those in row 8.

For the results in row 10 in Table 2-3, the no-tax proportion of bills and bonds, g_2, was increased by 5.0 percent. This caused bank i to increase its loan rate and thus lower its market share. The bank chose to hold 5.0 percent more in bills and bonds (8.5 more in value), which, with $FUNDS_{it}^e$ unchanged, caused LB_{it}^e to decrease by 8.5. The higher loan rate was the rate necessary to lead to a decrease in the expected demand for the bank's loans of this amount. The results in row 11, based on a 5.0 percent decrease in g_2, are essentially the opposite to those in row 10. In this case the bank chose to hold a slightly higher proportion of bills and bonds than the no-tax proportion as a result of the lower loan rate relative to the bill rate.

The results in Table 2-3 can be summarized briefly as follows. A bank is constrained in how much it can lend to households and firms (LB_{it}^e) and in the value of bills and bonds that it can purchase (VBB_{it}) by its expected level of funds ($FUNDS_{it}^e$). When $FUNDS_{it}^e$ increases, bank i lowers its loan rate, thus increasing LB_{it}^e, and increases VBB_{it}. The opposite happens when $FUNDS_{it}^e$ decreases. When either the bill rate for the current period increases or bank j's rate of the previous period increases, bank i increases its loan rate for the current period because it expects that bank j's loan rate for the current period will be higher than otherwise. The opposite happens when the rates decrease. When the demand for loans of the previous period increases, with no change in $FUNDS_{it}^e$, this also causes bank i to increase its loan rate for the current period in order to lower its expected market share. The opposite happens when the demand for loans of the previous period decreases. Because of the restriction that VBB_{it} + LB_{it}^e equals $FUNDS_{it}^e$, LB_{it}^e can increase, with $FUNDS_{it}^e$ unchanged, only at the expense of VBB_{it}, and vice versa. When the bill rate decreases relative to the loan rate, there is a tendency for the bank to substitute away from bills and bonds into expected loans, and vice versa.

2.6 THE CONDENSED MODEL FOR BANKS

The bank behavioral equations for the condensed model are presented in Table 2-4. In terms of notation, all i subscripts have been dropped from the variables, since for the condensed model there is only a bank sector rather than individual banks. Also, the loan rate for period t is now denoted RL_t rather than RB_{it}, and the level of savings deposits is denoted SD_t rather than SDB_{it}. Otherwise, the notation is the same for both the non-condensed and condensed models.

In Equation (1) in Table 2-4, $FUNDS_t^e$ is defined in exactly the same way as it is for the non-condensed model. Equation (2) determines the loan rate and is based on the results in Table 2-3. The coefficients were chosen to be consistent with the size of the reactions in Table 2-3. For example, a 5.0 percent increase in $FUNDS_{it}^e$ led to a 1.3 percent decrease in the loan rate in Table 2-3, and a 5.0 percent decrease in $FUNDS_{it}^e$ led to a 1.5 percent increase in the loan

Table 2-4. Bank Equations for the Condensed Model

(1)	$FUNDS_t^e = (1-g_1)(DDB_{t-1} - EMAXDD) + (SD_{t-1} - EMAXSD),$
(2)	$RL_t = 1.102(RL_{t-1})^{0.50}(FUNDS_t^e)^{-0.28}(g_2)^{0.10}(r_t)^{0.38}(LUN_{t-1})^{0.26},$
(3)	$VBB_t = 1.003\, g_2\, FUNDS_t^e (r_t)^{0.02}(RL_{t-1})^{-0.02},$
(4)	$LBMAX_t = FUNDS_t^e - VBB_t.$

rate. The average response was thus −1.4 percent, so that the elasticity of the loan rate with respect to $FUNDS_{it}^e$ is −1.4/5.0 = −0.28, which is the coefficient used for $FUNDS_t^e$ in Equation (2) in Table 2-4. The other coefficients were determined in a similar manner. The loan rate is a negative function of $FUNDS_t^e$ and a positive function of last period's loan rate, of g_2, of the bill rate, and of last period's unconstrained demand for loans.

Equation (3) determines the value of bills and bonds purchased by the bank sector. The equation is based on the results in Table 2-3 and states that the value of bills and bonds purchased deviates from the expected no-tax proportion ($g_2 FUNDS_t^e$) as a positive function of the bill rate and a negative function of last period's loan rate. The choice for the values of the constant terms in Equations (2) and (3) (1.102 and 1.003) will be discussed in Chapter Six.

Equation (4) is the same as for the non-condensed model. The bank sector is assumed to set the maximum value of loans that it will make in the period equal to the difference between its expected funds and the value of bills and bonds that it chooses to purchase.

NOTES

[a]Whenever an interest rate multiplies a stock in the model, the resulting interest revenue or interest payment, a flow, is assumed to be received or paid during the current period. For example, $RB_{it}LB_{it}$ in equation (2.2) is assumed to be the interest revenue received by bank *i* on its loans during period *t*.

[b]Since all expectations are made by bank *i*, no *i* subscript or superscript has been added to the relevant symbols to denote the fact that it is bank *i* making the expectation. The same procedure will be followed for firms and households below.

[c]In the programming for the non-condensed model, bank *i* was assumed to estimate the parameter α_3 in Equations (2.11) and (2.12) on the basis of its past observations of the correlation between changes in the aggregate unconstrained demand for loans and changes in the average loan rate. The exact procedure by which bank *i* was assumed to estimate α_3 is described in the Appendix. No *t* subscript is added to α_3 in the text, even though for the non-condensed model bank *i*'s estimate of α_3 will in general be changing from one decision period to the next.

[d]Although Equations (2.1) – (2.7) are written only for period *t*, they are also meant to hold for periods *t+1*, . . . ,*t+T* as well. In addition, an *e* superscript should be added to a variable when bank *i* only has an expectation of that variable. For example, Equation (2.2) should be written

$$\Pi B^e_{it} = RB_{it}LB^e_{it} + r_t VBILLB_{it} + BONDB_{it} - r_t SDB^e_{it} + (BONDB_{it}/R^e_{t+1} - BONDB_{it}/R_t), \quad (2.2)'$$

$$\Pi B^e_{it+k} = RB_{it+k}LB^e_{it+k} + r^e_{t+k}VBILLB_{it+k} + BONDB_{it+k} - r^e_{t+k}SDB^e_{it+k} + (BONDB_{it+k}/R^e_{t+k+1} - BONDB_{it+k}/R^e_{t+k}), k=1,2,\ldots,T. \quad (2.2)''$$

To conserve space, Equations (2.1) – (2.7) will not be written out in this expanded way, but the expansion in each case is straightforward.

Because Equations (2.1) – (2.7) are meant to hold for periods $t+1$ through $t+T$, some assumption has to be made about bank *i's* expectations of the future values of the tax parameters, d_1, d_2, and g_2, and the reserve requirement ratio, g_1. As is consistent with the practice in this study, bank i is assumed to expect the values of these parameters to remain unchanged over time. Similarly, firms and households are assumed to expect the values of the government tax parameters to remain unchanged over time. Because of these assumptions, t subscripts have not been added to the tax parameters and the reserve requirement ratio, although the government is, of course, free to change these parameters if it so desires.