11 Models with Rational Expectations

11.1 Introduction

The model considered in this chapter is one in which expectations of future values of endogenous variables appear as explanatory variables in the stochastic equations and the expectations are assumed to be rational in the Muth (1961) sense. This means that given a set of expectations of the exogenous variables, the expectations of the endogenous variables are equal to the model's predictions of these variables. The model (6.1) that was used for Chapters 6–10 must be modified for this chapter. The model will be written

$$f_i\left(y_t, y_{t-1}, \ldots, y_{t-p}, \underset{t-1}{E}\, y_t, \underset{t-1}{E}\, y_{t+1}, \ldots, \underset{t-1}{E}\, y_{t+h}, x_t, \alpha_i\right) = u_{it}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T, \tag{11.1}$$

where y_t is an n-dimensional vector of endogenous variables at time t, x_t is a vector of exogenous variables at time t, $\underset{t-1}{E}$ is the conditional expectations operator based on the model and on information through period $t-1$, α_i is a vector of unknown coefficients, and u_{it} is an error term. Compared to the notation in (6.1), x_t now includes only exogenous variables rather than both exogenous and lagged endogenous variables. As was the case for (6.1), the first m equations in (11.1) are assumed to be stochastic, with the remaining u_{it} ($i = m+1, \ldots, n$) identically zero for all t.

The key difference between (6.1) and (11.1) is the assumption that the expectations are rational. If they are not, but are instead, say, a function of the current and lagged values of a few variables, they can be substituted out of (11.1) to end up with a model like (6.1). This may introduce restrictions on the coefficients, but (6.1) already encompasses such restrictions. An example of this type of substitution is presented in Section 2.2.2, (2.1)–(2.3). In this case the expectation is only a function of the lagged values of the own variable. Another example is presented in Section 4.1.3, where expectations of price and wage inflation are assumed to be functions of a few lagged values.

An example of (11.1) is Sargent's model in Section 5.4, where the expectations variable $E_{t-1}p_t$ appears as an explanatory variable in the first two equations. Another example is presented later in this chapter in Section 11.7,

where the US model is modified to incorporate the assumption that there are rational expectations in the bond and stock markets.

The question of how to estimate and solve (11.1) is not easy. The next three sections are concerned with this question. A numerical method for solving the model for a given set of coefficients is discussed in Sections 11.2.1 and 11.2.2. A simple example is presented in Section 11.2.3 to motivate the method and to relate it to analytic techniques that have been used in previous research for solving and estimating rational expectations models. A numerical method for obtaining the full information maximum likelihood estimate of the coefficients is presented in Section 11.3. The possible use of stochastic simulation is discussed in Section 11.4, and the solution of optimal control problems for rational expectations models is considered in Section 11.5. Examples of using the methods are presented in Sections 11.6–11.8.

The solution method is an extension of the iterative technique used in Fair (1979d). In addition to dealing with serial correlation and multiple viewpoint dates, the extension involves an iterative procedure (called type III in the following discussion) designed to ensure numerical convergence to the rational expectations solution.

The estimation method is an extension to the nonlinear case of full information maximum likelihood techniques designed for linear rational expectations models, as described by Wallis (1980) and Hansen and Sargent (1980, 1981). Applications to particular economic problems are found in Sargent (1978) and Taylor (1980). The connection between the estimation problem considered in this chapter and the one considered by Hansen and Sargent appears in the f_i functions in (11.1), which for Hansen and Sargent would represent first-order conditions for the linear-quadratic optimization problem that they consider. Chow (1980) has proposed an alternative approach that leads to the same functional relationship between the structural parameters and the likelihood function as does the Hansen and Sargent approach.

Full information estimation techniques are particularly useful for rational expectations models because of the importance of cross-equation restrictions, where most of the testable implications of the rational expectations hypothesis lie. For linear models one can explicitly calculate a reduced form of model (11.1) in which the expectations variables are eliminated and nonlinear restrictions are placed on the coefficients. Under the assumption that the u_{it} are normally distributed, this restricted reduced form can be used to evaluate the likelihood function in terms of the structural coefficients. The maximum of the likelihood function with respect to the structural coefficients is found using some maximization algorithm like DFP.

For nonlinear models the reduced form cannot be calculated explicitly, but it can be calculated numerically. The estimation strategy here is to replace the calculation of the restricted reduced form in linear models with the numerical solution in nonlinear models. This permits one to evaluate the likelihood function in terms of the unknown structural coefficients much like in the linear case.

Although the solution and estimation methods described here should expand the range of empirical problems that can be approached using rational expectations, there is a limitation that may affect their general applicability. Because of computational costs, it is necessary in some applications to approximate the conditional expectations that appear in (11.1) by setting the future disturbances u_{it} equal to their conditional means in a deterministic simulation of the model. In nonlinear rational expectations models, the conditional expectations will involve higher-order moments of the u_{it} in addition to their means. (See Lucas and Prescott 1970, for example.) Although it is possible, as discussed in Section 11.4, to use stochastic simulation to obtain the conditional forecasts, this is computationally expensive. The results in Chapter 7 suggest that the bias introduced by using deterministic rather than stochastic simulation to solve models is small for typical macroeconometric models, and thus for many applications the use of stochastic simulation for rational expectations models is not likely to be needed. For other applications, however, the deterministic approximation may not be accurate, and stochastic simulation will be needed even though it is expensive.

With respect to (11.1), it should be noted that the model can include expectations of nonlinear functions of the endogenous variables. For example, if $y_{2t} = y_{1t}^2$, then the appearance of $\underset{t-1}{E}\, y_{2t}$ in one of the equations indicates that agents are concerned with the conditionally expected variance of y_{1t}. The model does not, however, include expectations based on current period *(t)* information. The incorporation of such variables does not cause difficulties for the solution of the model (as described below), but it does cause difficulties for estimation since the Jacobian of the transformation from the u_t to the y_t is altered.

11.2 A Solution Method

The numerical solution of (11.1) for a particular period s and for a given set of values of the α_i coefficients is considered in this section. The model without serial correlation of the errors is considered first, and then the modifications needed for the serial correlation case are discussed.

In the following discussion $\underset{t-1}{E}\, x_{t+j}$ will be used to denote the expected value of x_{t+j} based on information through period $t-1$. Both the actual realizations of x_t and the expected values are assumed to be known. If there are any exogenous variables that are not known but can be described by a known stochastic process, these are treated as endogenous and incorporated in the y_t vector. In this section, all simulations of the model are deterministic and are subject to the approximation mentioned in Section 11.1.

11.2.1 Models without Serial Correlation: The Basic Method

If one were given numerical values for the expected endogenous variables in (11.1) for all periods from s on, then it would be straightforward to solve the model for period s using the Gauss-Seidel iterative technique. The numerical method described here entails a series of iterations that converge from an arbitrary initial path of values for these expectations to a path that is consistent with the forecasts of the model itself. Let the initial set of values for the expected endogenous variables, $\underset{s-1}{E}\, y_{s+r}$, be represented as g_r, $r = 0, 1, \ldots$. Since in general the model will have no natural termination date, an infinite number of these values need to be specified in principle. In practice, however, only a finite number will be used in obtaining a solution with a given finite tolerance range. The initial values are required to be bounded: $|g_r| < M$ for every r, where M is not a function of r.

The solution method can be described in terms of five steps.

1. Choose an integer k, which is an initial guess at the number of periods beyond the horizon h for which expectations need to be computed in order to obtain a solution within a prescribed tolerance level δ. Set $\underset{s-1}{E}\, y_{s+r}$ equal to g_r, $r = 0, 1, \ldots, k + 2h$. For the purpose of describing the iterations, call these initial values $e_r(1,k)$, $r = 0, 1, \ldots, k + 2h$; the values at later iterations will then be called $e_r(i,k)$, $i > 1$.

2. Obtain a new set of values for $\underset{s-1}{E}\, y_{s+r}$, $r = 0, 1, \ldots, k + h$, by solving the model dynamically for y_{s+r}, $r = 0, 1, \ldots, k + h$. This is done by setting the disturbances to their expected values (usually zero), using the values $\underset{s-1}{E}\, x_s, \ldots, \underset{s-1}{E}\, x_{s+h+k}$ in place of the actual x's, and using the values $e_r(i,k)$ in place of $\underset{s-1}{E}\, y_{s+r}$. Call these new guesses $e_r(i + 1, k)$, $r = 0, 1, \ldots, k + h$. If the model is nonlinear, the solution for each period requires a series of Gauss-Seidel iterations. Call each of these a type I iteration.

3. Compute for each expectations variable and each period the absolute value of the difference between the new guess and the previous guess, that is, compute the absolute value of the difference between each element of the $e_r(i+1, k)$ vector and the corresponding element of the $e_r(i,k)$ vector for $r = 0, 1, \ldots, h+k$. If any of these differences are not less than a prescribed tolerance level (that is, if convergence has not been achieved), increase i by 1 and return to step 2. If convergence has been achieved, go to step 4. Call this iteration (performing steps 2 and 3), a type II iteration. (The type II tolerance level should be smaller than δ, which is the overall tolerance level. Similarly, the type I tolerance level should be smaller than the type II tolerance level.) Let $e_r(k)$ be the vector of the convergent values of a series of type II iterations ($r = 0, 1, \ldots, k+h$).

4. Repeat steps 1 through 3 replacing k by $k+1$. Compute the absolute value of the difference between each element of the $e_r(k+1)$ vector and the corresponding element of the $e_r(k)$ vector, $r = 0, 1, \ldots, h$. If any of these differences are not less than δ, increase k by 1 and repeat steps 1 through 4. If convergence has been achieved, go to step 5. Call this iteration (performing steps 1 through 4) a type III iteration. Let e_r be the vector of the convergent values of a series of type III iterations ($r = 0, 1, \ldots, h$).

5. Use e_r for $\underset{s-1}{E}\, y_{s+r}$, $r = 0, 1, \ldots, h$, and the actual values for x_t to solve the model for period s. This gives the desired solution, say $\hat{y}_s$, and concludes the solution method.

To summarize, the method just outlined iterates on future *paths* of the expected endogenous variables, $\underset{s-1}{E}\, y_{r+s}$. Starting from an initial guess at the path g_r, $r = 0, 1, 2, \ldots, k+2h$, the path is *extended* beyond $k+2h$ until further extensions do not affect the solution by more than δ.

Note with respect to step 3 that in the process of achieving type II convergence, the initial guesses $e_r(1,k)$, $r = k+h+1, \ldots, k+2h$, never get changed. These guesses are needed to allow the model to be solved through period $s+h+k$. Also note that when one is repeating steps 1 through 3 for $k+1$, it may be possible to speed convergence by using some information from iteration k. The most obvious thing to do is to use as initial guesses $e_r(1,k+1) = e_r(k)$, $r = 0, 1, \ldots, k+h$. The values g_r would then be used for $e_r(1, k+1)$, $r = k+h+1, \ldots, k+2h+1$.

Computational costs for the method are determined by the total number of passes through the model required for convergence. A pass is simply a single evaluation of the LHS endogenous variables in terms of the RHS variables. Let N_1 be the number of type I iterations required for convergence, and let N_2

be the number of type II iterations required for convergence. Then the number of passes through the model required for one type III iteration is given by the product of the number of passes for one type II iteration, $N_1 \times (h + k + 1)$, and the number of type II iterations required for convergence, N_2. The total number of passes through the model to obtain type III convergence is given by the sum of this expression from k to $k + N_3 - 1$, where N_3 is the number of type III iterations required for convergence. In other words, the number of passes through the model required by type III convergence is approximately

$$\sum_{q=k}^{k+N_3-1} [N_2 \times N_1 \times (h + q + 1)].$$

This formula is only approximate because it is based on the assumption of the same number of type I iterations for each period and the same number of type II iterations for each type III iteration. In practice this is usually not the case.

Two points about the solution method should be noted. First, it can be easily modified to handle the case in which the expectations are based on information through period s rather than through period $s - 1$: one just replaces E_{s-1} by E_s everywhere. Second, if the expectations horizon is infinite ($h = \infty$), then it must be truncated first. For most models the error introduced by this truncation for reasonably large values of h is likely to be small. A large value of h means, of course, that a large number of calculations are required per type II iteration, and thus in practice there may be a trade-off between truncation error and computational cost.

For a general nonlinear model there is no guarantee that any of the iterations will converge. If convergence is a problem, it is sometimes helpful to damp the successive solution values. "Damping" means to take the value of a variable at, say, the start of iteration n to be some fraction of the difference between the value actually computed on iteration $n - 1$ and the value used at the start of iteration $n - 1$. (See the discussion of damping in Section 7.2.)

In special cases a problem may have terminal conditions. If, say, the values $E_{s-1} y_{s+r}$, $r = k + h + 1, \ldots, k + 2h$, are known, then the present method gives the correct answer after type II convergence. No type III convergence tests are needed because the values for periods $s + k + h + 1$ through $s + k + 2h$ are known. Cases with terminal conditions are referred to as two-point boundary value problems. They have been used to study rational expectations models when one can approximate the terminal conditions with steady-state values, which may be derived in certain situations. (See Lipton et

al. 1982, who use a "multiple shooting" method to solve the two-point boundary value problem.) The approximation that comes from equating the terminal conditions with the steady-state values does not arise with the present method. Moreover, the method does not require that one compute steady-state values beforehand.

One final point about the solution method should be noted. If the model either has no exogenous variables or if the actual values of the exogenous variables are used for all periods, the solution values of the expectations—$\underset{s-1}{E}\, y_{s+r}$, $r = 0, 1, \ldots, h$—are the final predicted values of the model. This means that $\hat{y}_s$ in step 5 is simply $\underset{s-1}{E}\, y_s$, and therefore step 5 does not have to be done. It also means that if a dynamic simulation is to be run for, say, periods s through $s + q$, the model only needs to be solved once in the above manner (for period s) to get all the predicted values if q is less than or equal to h.

For purposes of the following discussion, the method presented in this section will be called the "basic method."

11.2.2 Models with Serial Correlation

Forecasting and Policy Applications

The case of first-order serial correlation is considered in this section:

$$u_{it} = \rho_i u_{it-1} + \epsilon_{it}, \qquad i = 1, \ldots, n, \tag{11.2}$$

where the ρ_i are serial correlation coefficients. The solution method is first modified for applications in which there are enough data prior to the solution period s to permit calculation of the solution values with only a negligible effect of the errors prior to period $s - 1$. This situation is likely to occur in forecasting or policy applications, where a large sample prior to the simulation period is usually available. The method is then modified for estimation applications, where sufficient prior data are generally not available.

First note that (11.1) and (11.2) can be combined to yield

$$f_i(y_t, y_{t-1}, \ldots, y_{t-p}, y_{t-p-1}, \underset{t-1}{E}\, y_t, \underset{t-1}{E}\, y_{t+1}, \ldots, \underset{t-1}{E}\, y_{t+h}, \underset{t-2}{E}\, y_{t-1}, \underset{t-2}{E}\, y_t, \ldots, \underset{t-2}{E}\, y_{t+h-1}, x_t, x_{t-1}, \alpha_i, \rho_i) = \epsilon_{it}, \quad i = 1, \ldots, n, \tag{11.3}$$

where the ρ_i can be thought of as structural coefficients. For solution purposes the important difference between (11.1) and (11.3) is the addition in (11.3) of

an extra viewpoint data ($t-2$). This requires an additional type of iteration, denoted type IV.

If one were given values for the expectations with viewpoint date $s-2$, then (11.3) could be solved using the basic solution method in Section 11.2.1. The expectations with viewpoint date $s-2$ could be obtained by solving the model one period earlier at time $s-1$, but this in turn would require values for the expectations with viewpoint data $s-3$, and so on. By working backward in this way, however, it is possible to ensure that these initial values have negligible influence on the current period s.

The procedure is as follows.

(a) Choose an integer j, which is an initial guess at the number of periods before period s for which the model needs to be solved in order to achieve the prescribed tolerance level. Set $\underset{s-j-2}{E}\, y_{s-j-1+r}$, $r=0, 1, \ldots, h$, to an initial set of values. (As with the basic method, the initial guesses are required to be bounded.)

(b) Given the values from (a), solve the model for period $s-j$ using the basic method. For this solution the viewpoint date for the expectations for x_{s-j} and beyond is $s-j-1$. Actual values are used for x_{s-j-2}. The solution yields values for $\underset{s-j-1}{E}\, y_{s-j+r}$, $r=0, 1, \ldots, h$.

(c) Given the expectations with viewpoint date $s-j-1$ from (b), solve the model for period $s-j+1$ using the basic method. For this solution the viewpoint date for the expectations for x_{s-j+1} and beyond is $s-j$. Actual values are used for x_{s-j-1}. This solution yields values for $\underset{s-j}{E}\, y_{s-j+1+r}$, $r=0, 1, \ldots, h$. Continue this procedure (using the basic method to solve for the next period, given the solved-for expectations from the previous period) through period s. The solution for period s yields values for $\underset{s-1}{E}\, y_{s+r}$, $r=0, 1, \ldots, h$.

(d) Increase j by 1 and repeat (a) through (c). This yields new values for $\underset{s-1}{E}\, y_{s+r}$, $r=0, 1, \ldots, h$. Compare these values to the values obtained by using the smaller j. If any new value is not within the prescribed tolerance level of the old value, increase j by 1 and repeat steps (a) through (c). Keep doing this until convergence is reached. Call this iteration (performing steps a through c) a type IV iteration. (The tolerance level for the type IV iterations should be greater than δ, the tolerance level for the type III iterations.)

(e) After type IV convergence one has final values of $\underset{s-1}{E}\, y_{s+r}$ and $\underset{s-2}{E}\, y_{s-1+r}$, $r=0, 1, \ldots, h$. Use these values and the actual values of x_s and x_{s-1} to solve the model for period s.

Each type IV iteration requires solving the model for $j+1$ starting points (that is, achieving type III convergence $j+1$ times). The serial correlation case is thus considerably more expensive than the nonserial correlation case when one is solving the model for one period. However, no additional type IV iterations are required for solving the model for periods later than s, once the solution for period s has been obtained. The predictions with viewpoint date $s-1$ are known after solving for period s, for example, and they can be used in solving for period $s+1$.

It should be emphasized that type IV iterations can handle problems that are more general than the case of first-order autoregressive errors. In particular, the expectations variables with viewpoint dates $t-2$ need not arise solely from the presence of autoregressive errors, and there can be more than two viewpoint dates. If, say, viewpoint date $t-3$ were also included in the model, the only change in the procedure would be the addition of initial guesses for E_{s-j-3} values in step (a). One would merely need to keep track of three sets of expectations instead of two as the solutions proceeded from period $s-j$ to period s.

Estimation Applications

Type IV iterations require sufficient data prior to the solution period that the initial guesses have a negligible effect on the solution. In most estimation problems one would not want to lose as many observations from the beginning of the sample as would be required for type IV convergence. Fortunately, there is a way around this problem, which is based on an assumption that is usually made when one is estimating multiple equation models with moving average residuals. This assumption is that the last presample *uncorrelated* error is zero; in particular that $\epsilon_{is-1}=0$ in (11.2) when one is solving for period s. As before, the case of first-order serial correlation is considered; generalization to higher orders is fairly straightforward. The method requires data for period $s-1$. (Data before period $s-1$ will be needed if there are lagged endogenous or lagged exogenous variables in the model. It is implicitly assumed here that sufficient data for the lagged variables are available for the solution for period $s-1$.) Rather than first transforming (11.1) into (11.3), the method works directly with (11.1), treating (11.2) as another set of equations.

If u_{is-2} were known, then (11.1) could be solved for period $s-1$ and all subsequent periods using the basic method and the fact that $E_{s-2} u_{is+r}=\rho_i^{(r+2)}u_{is-2}$. In other words, in the dynamic simulations that underlie the basic method, one would use $\rho_i^{(r+2)}u_{is-2}$ on the RHS of (11.1). The

problem then becomes one of choosing an appropriate value for u_{is-2}. This is where the assumption about ϵ_{is-1} comes in. The idea is to choose u_{is-2} in such a way that when the model is solved for period $s-1$, it generates a value of $\epsilon_{is-1}=0$; that is, $u_{is-1}=\rho_i u_{is-2}$. The rationale for this choice is simply that 0 is the unconditional mean of ϵ_{is-1}, and thus the actual value is likely to be relatively close to this value.

An iterative procedure for choosing u_{is-2} so that $\epsilon_{is-1}=0$ can be described as follows (note that each calculation is performed for each equation $i=1, \ldots, m$).

(i) Guess values for the error terms u_{is-2}.

(ii) Given the values from (i), solve the model for period $s-1$ using the basic method. Note that $\underset{s-2}{E}\, u_{s+r}$ is set to $\rho_i^{(r+2)}u_{is-2}$ in calculating the predicted values.

(iii) Given the predicted value of y_{is-1} ($\hat{y}_{is-1}$) from step (ii), calculate $\hat{\epsilon}_{is-1}=y_{is-1}-\hat{y}_{is-1}$ and $\hat{u}_{is-1}=\rho_i u_{is-2}+\hat{\epsilon}_{is-1}$, where u_{is-2} is the initial guess. If $\hat{\epsilon}_{is-1}$ is not within a prescribed tolerance level of 0, then convergence has not been reached, (that is, the solution is not consistent with the assumption that $\epsilon_{is-1}=0$).

(iv) If convergence is not reached in (iii), set the new value of u_{is-2} equal to $\hat{u}_{is-1}/\rho_i$ and do (ii) and (iii) over for these new values. Repeat this until convergence is reached.

(v) Using the converged iterate u_{is-2}, compute $u_{is-1}=\rho_i u_{is-2}$. Given these values, solve for period s using the basic method, where in this case $\underset{s-1}{E}\, u_{s+r}=\rho_i^{(r+1)}u_{is-1}$ is used in calculating the predicted values. This completes the solution for period s.

Once the solution for period s has been obtained, the solutions for periods $s+1$ and beyond do not require further iterations from those used by the basic method. The reason for this is that the forecasts with viewpoint date $s-1$ are known after solving for period s.

11.2.3 A Simple Example

The conditions under which the solution method just presented will converge from an arbitrary set of initial guesses to the rational expectations solution are examined for a simple linear model in this section. The aim is to motivate the method and relate it to existing analytic techniques.

A scalar linear version of (11.1) with serial correlation is given by

$$y_t = \alpha \underset{t-1}{E}\, y_{t+1} + \gamma \underset{t-1}{E}\, x_t + u_{1t}, \tag{11.4}$$

$$x_t = \lambda x_{t-1} + \epsilon_{2t}, \tag{11.5}$$

$$u_{1t} = \rho u_{1t-1} + \epsilon_{1t}, \tag{11.6}$$

where α, γ, λ, and ρ are scalar parameters and $(\epsilon_{1t}, \epsilon_{2t})$ is a serially uncorrelated vector. It is assumed that $|\lambda| < 1$ and $|\rho| < 1$. Equations (11.4) and (11.5) correspond to (11.1) when the exogenous variable x_t is assumed to follow a known stochastic process, and (11.6) corresponds directly to the autoregressive error assumption made in (11.2).

The rational expectations solution of (11.4) through (11.6) in period s is given by

$$\begin{aligned} \underset{s-1}{E}\, y_s &= \sum_{i=0}^{\infty} \alpha^i \lambda^{i+1} x_{s-1} + \sum_{i=0}^{\infty} \alpha^i \rho^{i+1} u_{1s-1} \\ &= \frac{\gamma\lambda}{1-\alpha\lambda} x_{s-1} + \frac{\rho}{1-\alpha\rho} u_{1s-1}. \end{aligned} \tag{11.7}$$

(See Hansen and Sargent 1981 and Taylor 1980 for discussion of an analytic solution method.) Note that the last equality in (11.7) requires that $|\alpha\lambda| < 1$ and $|\alpha\rho| < 1$, which will be satisfied if $|\alpha| < 1$. The objective is to show that the numerical solution method generates the same solution value as that given in (11.7). For now take u_{1s-1} as given; a procedure for calculating u_{1s-1} is described subsequently. Recall that $e_r(i,k)$ is the guess of $\underset{s-1}{E}\, y_{s+r}$ on type II iteration i and type III iteration k. Each type III iteration is started with an initial set of guesses $e_r(1,k)$, $r = 0, 1, \ldots, k+2$ ($h = 1$ in this example). The aim is to show that $\lim_{i,k \to \infty} e_0(i,k)$ equals the RHS of (11.7).

For a fixed k the type II iterations can be described by the set of equations

$$e_r(i+1, k) = \alpha e_{r+1}(i,k) + \gamma\lambda^r x_{s-1} + \rho^r u_{1s-1}, \tag{11.8}$$

where $r = 0, 1, \ldots, k+1$. By repeated substitution

$$\begin{aligned} e_0(k+3, k) = (\alpha)^{k+2} e_{k+2}(1, k) &+ \gamma\lambda \sum_{h=1}^{k+1} (\alpha\lambda)^h x_{1s-1} \\ &+ \rho \sum_{h=1}^{k+1} (\alpha\rho)^h u_{1s-1}, \end{aligned} \tag{11.9}$$

which is the converged iterate of the type II iterations for a fixed k. Equation (11.9) is not equal to the RHS of (11.7). However, if $|\alpha| < 1$, then the limit of

$e_0(k+3, k)$ as $k \to \infty$ is equal to the RHS of (11.7). This motivates the requirement that the initial values $e_{k+2}(1,k) \equiv g_{k+2}$ are bounded, and it shows that type III iterations converge to the rational expectations solution. Note that in this model the solution is independent of all g_r values. Given that the g_r values are bounded, type III iterations ensure convergence to the correct answer.

Note that the condition for this convergence ($|\alpha| < 1$) is identical to the condition needed to obtain a unique solution in rational expectations models (see Taylor 1977). This suggests that the numerical method will converge in the class of rational expectations models for which the uniqueness conditions hold, although a general proof is still open.

This example will now be used to illustrate the relationship between the procedure described in Section 11.2.2 (designed to choose initial conditions for estimation applications) and the conditional maximum likelihood estimates of linear ARMA models.

Substituting (11.7) into (11.4) results in

$$y_t = \frac{\gamma\lambda}{1-\alpha\lambda} x_{t-1} + \frac{\rho}{1-\alpha\rho} u_{1t-1} + \epsilon_{1t}. \tag{11.10}$$

Subtracting the lagged value of (11.10) multiplied by ρ from (11.10) results in the "quasi-differenced" expression

$$y_t = \rho y_{t-1} + \frac{\gamma\lambda}{1-\alpha\lambda}(x_{t-1} - \rho x_{t-2}) + \frac{\alpha\rho^2}{1-\alpha\rho}\epsilon_{1t-1} + \epsilon_{1t}, \tag{11.11}$$

which when combined with (11.5) gives a two-dimensional vector ARMA(2,1) model with nonlinear constraints on the parameters. For estimation of the parameters of this ARMA model it is necessary to calculate the residuals $(\epsilon_{1t}, \epsilon_{2t})$ in terms of the data and the parameters. For "conditional" maximum likelihood estimates, this calculation is started by setting $\epsilon_{1s-1} = 0$ and taking y_{s-1}, x_{s-1}, and x_{s-2} as given, where s is the beginning of the estimation period. The residual ϵ_{1s} is then computed by subtracting (11.11) with these values from the actual observation y_s. The residuals for later periods are calculated recursively using this computed residual ϵ_{1s}.

The procedure described in Section 11.2.2 is designed to calculate these "conditional" residuals numerically for linear as well as nonlinear models. This can be illustrated by showing that

$$\hat{y}_s = \rho y_{s-1} + \frac{\gamma\lambda}{1-\alpha\lambda}(x_{s-1} - \rho x_{s-2}) \tag{11.12}$$

when the value u_{1s-1} in (11.7) is chosen according to the procedure outlined in steps (i) through (v) in Section 11.2.2. It is known from (11.7) that the basic numerical solution method will generate

$$\hat{y}_{s-1} = \frac{\alpha\lambda}{1-\alpha\lambda} x_{s-2} + \frac{\rho}{1-\alpha\rho} u_{1s-2} \tag{11.13}$$

when applied in period $s-1$, as indicated in step (ii). Iterating steps (iii) and (iv) will yield a converged iterate of u_{1s-2} that has the property that $y_{s-1} - \hat{y}_{s-1} \equiv \epsilon_{1s-1} = 0$ to within the tolerance level. From (11.13) this value of u_{1s-2} is given by

$$u_{1s-2} = \frac{1-\alpha\rho}{\rho}\left(y_{s-1} - \frac{\gamma\lambda}{1-\alpha\lambda} x_{s-2}\right) \tag{11.14}$$

and therefore

$$u_{1s-1} = \rho u_{1s-2} = (1-\alpha\rho)\left(y_{s-1} - \frac{\gamma\lambda}{1-\alpha\lambda} x_{s-2}\right). \tag{11.15}$$

Substituting (11.15) into (11.17) yields (11.12), which is what is to be shown. Note that when analytic techniques can be used, it is trivial to choose u_{1s-2} according to (11.14), but when the solutions are calculated numerically, it is necessary to search for the value u_{1s-2} that gives $\epsilon_{1s-1} = 0$.

11.3 FIML Estimation

11.3.1 Evaluating and Maximizing the Likelihood Function

FIML estimates of the coefficients are obtained by maximizing L in (6.33), which is repeated here:

$$L = -\frac{T}{2}\log|S| + \sum_{t=1}^{T}\log|J_t|. \tag{6.33}$$

S is the $m \times m$ matrix whose ij element is $\frac{1}{T}\sum_{t=1}^{T} u_{it}u_{jt}$, and J_t is the $n \times n$ Jacobian matrix whose ij element is $\partial f_i/\partial y_{jt}$. Because the expectations in (11.1) are based only on information through period $t-1$ (and thus not on y_{jt}), the derivatives of the expectations with respect to the y_{jt} $(j = 1, \ldots, n)$ are zero. The expectations are thus like the exogenous variables with respect to the Jacobian calculations.

Given the solution method in Section 11.2, it is straightforward to compute

L for a given value of α for rational expectations models. If there is no serial correlation, then for a given value of α one can solve for $\underset{s-1}{E}\, y_s$, $\underset{s-1}{E}\, y_{s+1}$, . . . , $\underset{s-1}{E}\, y_{s+h}$ for $s = 1, 2, \ldots, T$ using the solution method. These values can then be used in conjunction with the y and x data to compute values of u_{is} ($s = 1, 2, \ldots, T$) and thus the matrix S. The Jacobian determinants can be computed in the usual way, thereby completing the determination of L. The extra work involved in the calculation of L for rational expectations models thus consists of using the solution method to compute the expected values for each of the T viewpoint dates. For models without rational expectations none of these calculations are needed. Given this extra work, however, FIML estimates can be obtained in the usual way by maximizing L numerically with respect to α. For small models an algorithm like DFP may be sufficient to maximize L, but for other models the Parke algorithm is likely to be needed.

When the u_{it} follow a first-order autoregression process, only one main change to the procedure given above is necessary. In this case steps (i) through (iv) in Section 11.2.2 are needed to calculate the expected values for the first sample point (say, period 2). Given these expected values, which have viewpoint date 1, the expected values for period 3 can be obtained using the solution method. These expected values can then be used in the calculation of the expected values for period 4, and so on through the end of the sample period. The only extra work in the serial correlation case pertains to the first sample point. Numerical maximization in this case is with respect to both the structural coefficients and the serial correlation coefficients.

11.3.2 A Less Expensive Method for Maximizing the Likelihood Function

The procedure in Section 11.3.1 is expensive because many evaluations of L are needed in the process of maximizing the likelihood function, and the model must be solved T times for each evaluation of L. This requires a very large number of passes through the model for a given estimation problem. In this section a way of modifying the estimation method is considered that requires fewer calls to the solution method. This modification is as follows.

(A) Given the initial value of α, solve for $\underset{s-1}{E}\, y_s$, $\underset{s-1}{E}\, y_{s+1}$, . . . , $\underset{s-1}{E}\, y_{s+h}$ for $s = 1, 2, \ldots, T$ using the solution method. This requires doing steps 1–5 in Section 11.2.1 T times. Call the solution values from this step the "base" values.

(B) Perturb each coefficient (one at a time) from its initial value and use the solution method to get a new set of solution values. From these values and

the base values, calculate numerically the derivatives of the expectations with respect to the coefficients. This step requires doing steps 1–5 T times for each coefficient.

(C) In the procedure that calculates L for a given value of α, use the base values and the derivatives to calculate new expected values for each new value of α. This eliminates the need to use the solution method in computing new values of L.

(D) Once the maximization algorithm has found the value of α that maximizes L, compute a new set of base values using the new value of α and a new set of derivatives. Given the new derivatives, use the maximization algorithm again to find the value of α that maximizes L. Keep doing this until the successive estimates of α from one use of the maximization algorithm to the next are within a prescribed tolerance level.

The advantage of this modification is that once the problem is turned over to the maximization algorithm, the solution method is no longer needed. The use of the base values and derivatives in the calculation of L is very inexpensive relative to the use of the solution method, and given that algorithms require many calculations of L, this modification is likely to result in a considerable saving of time. There is, of course, no guarantee that the procedure will converge. If the expectations are not a well-behaved function of α, then computing the derivatives at a given point may not be very helpful. It may be, in other words, that using the base values and derivatives to calculate new expected values yields values that are far from the (correct) values that would be computed by the solution method.

Once the estimates have been obtained, the covariance matrix in (6.34) can be calculated by taking numerical derivatives of L with respect to α (at the optimum). It may be possible to use the derivatives of the expectations with respect to α in the calculation of the values of L. This would allow the covariance matrix to be computed without using the solution method.

For the serial correlation case one must also calculate in step (B) the derivative of $\hat{u}_{is-1}$ with respect to α (for each i), where s is the first sample point. $\hat{u}_{is-1}$ is a function of α, and therefore if steps (i)–(iv) are to be bypassed in the calculation of L, the derivative of $\hat{u}_{is-1}$ with respect to α must also be calculated and used.

11.4 Solution and Estimation Using Stochastic Simulation

The use of stochastic simulation to estimate and solve rational expectations models is discussed in this section. The case of (11.1) with no serial correlation will be considered.

Consider first the problem of solving a rational expectations model. Suppose that both the α coefficients in (11.1) and S are known, where S is the covariance matrix of the disturbances u_{it}. Assume that the u_{it} are normally distributed. The solution procedure is modified as follows. First, the expected values computed in step 2—$\underset{s-1}{E}\, y_{s+r}$, $r = 0, 1, \ldots, k+h$—are computed by stochastic rather than deterministic simulations. Instead of setting the disturbances to their expected values and solving once, one solves the model for many different trials. Each trial consists of a set of draws of the disturbances u_{is+r}, $r = 0, 1, \ldots, k+h$, from the $N(0,S)$ distribution (assuming the expected values of all the disturbances are zero). Each expected value is computed as the average across all the trials. Second, the final solution value $\hat{y}_s$ computed in step 5 is also computed by a stochastic rather than a deterministic simulation. In this case only draws of the disturbances for period s are needed.

Stochastic simulation can also be used to obtain FIML estimates of the coefficients. In contrast to the deterministic case, however, the likelihood function cannot be "concentrated" as it is in (6.33). In the fully stochastic case, changes in S affect the solution of the model and thereby the computed residuals. Instead, one works directly with the "unconcentrated" (log) likelihood function, which except for a constant can be written

$$L^* = \sum_{t=1}^{T} \log|J_t| - \frac{T}{2}\log|S| - \frac{1}{2}\sum_{t=1}^{T} u_t' S^{-1} u_t, \tag{11.16}$$

where $u_t = (u_{1t}, \ldots, u_{mt})'$. FIML estimates can be obtained by maximizing L^* with respect to the parameters (α,S). Each evaluation of L^* for a given set of values of α and S requires computing the expected values, $\underset{t-1}{E}\, y_{t+r}$, $r = 0, 1, \ldots, k+h$, by means of stochastic simulation, where each trial consists of draws of the disturbances from the $N(0,S)$ distribution. The expected values are computed for each sample point $t = 1, \ldots, T$, which then allows u_t to be computed for each point. The determinants of the J_t can be obtained, and thus the function L^* can be evaluated in terms of the parameters (α,S). Nonlinear maximization routines can then be used to maximize L^*.

Because this estimation procedure requires maximization over the $(m+1)m/2$ independent elements of S in addition to the elements of α and because of the stochastic simulation costs, the method is likely to be extremely expensive in practice. Given this, experiments with the method on small representative nonlinear models would be useful to try to gauge how much accuracy is likely to be gained by using stochastic simulation.

11.5 Solution of Optimal Control Problems for Rational Expectations Models

The method for solving optimal control problems in Section 10.2 merely requires the ability to solve the model for a given set of values of the control variables. Given this, the problem is turned over to a maximization algorithm like DFP to find the optimum. The method in Section 11.2 provides the ability to solve rational expectations models, and thus optimal control problems can be solved for these models by using this solution method within the context of the overall method in Section 10.2.

Since rational expectations models are forward-looking, future values of the control variables affect current decisions, and therefore more values of the control variables have to be determined in this case than in the standard case. Values of the control variables must be chosen far enough into the future so that adding another future period has a negligible effect on the solutions for the actual control problem. The solution method in Section 11.2 ensures that the predicted values in the last future period have a negligible effect on the predicted values for the current period, and thus the requirement for the optimal control problem is merely to choose the number of control values that are required by the solution method in the course of solving the model.

There is a potential problem of time inconsistency in solving optimal control problems for rational expectations models, which has been pointed out by Kydland and Prescott (1977). Consider a deterministic setting, and assume that a control problem has been solved using the above procedure for periods 1 through T. This yields optimal values $z_1^*, z_2^*, \ldots, z_T^*$. Now wait for one period, and consider the solution of the problem at the beginning of period 2. Since the setting is deterministic, nothing unexpected has happened, and therefore one might think that the same optimal values $z_2^*, \ldots, z_T^*$ would be determined. If the model is forward-looking, this is not necessarily the case, and when it is not, the optimal policy is said to be time-inconsistent. The model does not have to be a rational expectations model in order for this problem to arise; it only needs to have the property that future values of the control variables affect current decisions.

The problem of time inconsistency does not mean that the above solution of the control problem is not optimal. It is optimal if it is believed and carried out. The problem is that the policymakers have an incentive to do something different in the future, and therefore agents may not believe that the original plan will be carried out. If it is not possible for the policymakers to convince agents that the plans will be carried out, other policies may be better. Even in

this case, however, it is of some interest to solve the control problem in the above manner in order to have a benchmark to which other policies can be compared.

11.6 Results for a Small Linear Model

11.6.1 Model without Serial Correlation

For purposes of testing the solution and estimation methods, a small linear model has been analyzed. This model can be solved and estimated using existing linear techniques, and thus it provides a useful check for the nonlinear methods. The model is a version of the wage-contracting model in Taylor (1980). It can be represented as

$$(11.17)\quad y_{1t} = \alpha_{11}y_{1t-1} + \alpha_{12}y_{1t-2} + \alpha_{13}\underset{t-1}{E}\, y_{1t+1} + \alpha_{14}\underset{t-1}{E}\, y_{1t+2} + \alpha_{15}\underset{t-1}{E}\, y_{2t} + \alpha_{16}\underset{t-1}{E}\, y_{2t+1} + \alpha_{17}\underset{t-1}{E}\, y_{2t+2} + u_{1t},$$

$$(11.18)\quad y_{2t} = \alpha_{21}y_{1t} + \alpha_{22}y_{1t-1} + \alpha_{23}y_{1t-2} + u_{2t},$$

with restrictions $\alpha_{11} = \alpha_{13} = \frac{1}{3}$, $\alpha_{12} = \alpha_{14} = \frac{1}{6}$, $\alpha_{15} = \alpha_{16} = \alpha_{17}$, $\alpha_{21} = \alpha_{22} = \alpha_{23}$. There are two free coefficients to estimate, α_{15} and α_{21}. The data for this model were generated by simulating the model using normally distributed serially independent errors with zero correlation between equations. Values of α_{15} and α_{21} of .0333333 and $-.333333$ were used for this purpose.

The model was first solved and estimated using the technique described in Taylor (1980), which is based on a factorization procedure that calculates a restricted ARMA version of the model. The ARMA version is used for the likelihood function calculations. Because of its small size, the model does not require the use of the Parke algorithm for the FIML estimation, so the DFP algorithm was used. Using a sample of 50 observations, the estimated coefficients were $\hat{\alpha}_{15} = .02601$ and $\hat{\alpha}_{21} = -.3916$, with t-statistics of 1.18 and 6.33, respectively. Each evaluation of the likelihood function took about .004 seconds on an IBM 360/91 at Columbia University using this factorization technique. The DFP algorithm required 90 function evaluations starting from the true values (.0333333 and $-.333333$).

The model was next solved using the method in Section 11.2. The model was solved for all 50 observations, and the value of the likelihood function was computed. When evaluated at the same coefficient values, the method gave the same value of the likelihood function as did the factorization

TABLE 11-1. Computational summary of the likelihood function evaluations for the small linear model

Model with no serial correlation (h = 2):

1. The initial value of k was taken to be 15. Type III convergence was almost always achieved after 2 iterations (i.e., for k = 16).
2. The average number of Type II iterations per Type III iteration was about 15.
3. Given the expectations, the model is recursive, so only one Type I iteration was needed for convergence each period. The average length of the simulation period for a Type II iteration was 18.5 periods, so the total number of Type I iterations for the solution for the 50 observations was about (50 obs.) × (1 Type I iteration) × (15 Type II iterations) × (18.5 periods per Type II iteration) × (2 Type III iterations) = 27750.
4. One Type I iteration requires about 10 multiplications and 7 additions.

Model with serial correlation (h = 2):

1. Call the first period of the sample period, period s. Steps i)-iv) were first used to calculate u_{1s-1}. The initial guess for u_{1s-2} was zero. A damping factor of .25 was used. Convergence took 18 iterations to obtain u_{1s-1}. For these calculations the initial value of k was taken to be 15, and Type III convergence was always achieved after 2 iterations. The average number of Type II iterations per Type III iteration was about 15. Given the expectations, the model is recursive, and so only one Type I iteration was needed for convergence each period. The number of passes for these calculations was thus about (18 iterations) × (1 Type I) × (15 Type II) × (18.5 periods per Type II) × (2 Type III) = 9990.
2. Given u_{1s-1}, step v) was used to solve for period s. This required 21 Type II iterations. The starting values for this step were the values computed in 1., and convergence was achieved after 1 Type III iteration. The number of passes for this step was thus (21 Type II) × (18 periods per Type II) = 378.
3. Given the solution for period s, the calculations for the remaining 49 periods are essentially the same as those above for the model with no serial correlation. These calculations thus required about 49 × 1 × 15 × 18.5 × 2 = 27195 passes.
4. The total number of passes through the model was thus about 9990 + 378 + 27195 = 37563.

technique, which serves as a useful check on both procedures. The details of the iterations of the method when solving the model are summarized in the upper section of Table 11-1. A total of about 27,750 passes through the model were required for one function evaluation, which is estimated to take about 1 second on an IBM 360/91. This is about 250 times slower than the factorization technique. (The actual computations were done on a computer at Yale University, and the estimated time for the IBM 360/91 is only approximate.)

Had the same DFP program been used to maximize the likelihood function

as was used for the factorization technique, the same 90 function evaluations would have been required to find the maximum. The reason for this is that the solution method and the factorization technique give the same value of the likelihood function for the same set of coefficient values, and this is all the information that the DFP algorithm takes from the methods. The total time needed to estimate the model would thus be about 90 seconds. The DFP calculations were not repeated, but instead an attempt was made to maximize the likelihood function using the less expensive method discussed in Section 11.3.2. These calculations will now be described.

The calculations using the less expensive method are summarized in Table 11-2. Using the true values of the coefficients as starting values, the model was first solved for each of the 50 observations. As noted in Table 11-1, this requires about 27,750 passes through the model. The model was then solved two more times to calculate the derivatives of the expectations with respect to the two coefficients. The problem was then turned over to the DFP algorithm. The computer program of the DFP algorithm used here was different from the program used above for the factorization technique, and the performance of the algorithm for a given problem does vary across programs. The following results thus differ in two respects from the results using the factorization technique: the derivatives are used in one case but not in the other, and the computer programs differ. It is not possible to say which of these factors is more important regarding the performance of the DFP algorithm, but this is not of great concern here. The question of interest is whether the use of the derivatives results in the optimum being found. (The program of the DFP algorithm used here is also not the one that I wrote and used for the results in Section 10.4. The work for the present section was done before I wrote the DFP program that is now part of the overall Fair-Parke program.)

As indicated in Table 11-2, the first DFP iteration required 45 calls to the subroutine that calculates L for a given value of the coefficient vector. Convergence was essentially achieved after the first iteration. The program was allowed to run for three more iterations, where for each iteration the model was solved three times: once to get the base values and twice more to get the derivatives. The results in Table 11-2 show that the use of the derivatives provides a close approximation to the "true" value of L obtained by solving the entire model. Given that the DFP algorithm required 45 evaluations of L (for the first iteration), the use of the derivatives saved a considerable amount of time. The derivatives were also used in the calculation of the covariance matrix after the optimum was reached.

TABLE 11-2. Results of estimating the small linear model using the less expensive method

	$\hat{\alpha}_{15}$	$\hat{\alpha}_{21}$	Value of L using expected values computed from derivatives	Value of L using expected values computed from steps 1-4	Number of times L was computed by the DFP algorithm
Initial values	.0333333	-.333333		508.6022686	
Iteration 1	.0252715	-.391654	509.0471277	509.0460742	45
Iteration 2	.0260208	-.391609	509.0466651	509.0466725	39
Iteration 3	.0260044	-.391616	509.0466727	509.0466724	20
Iteration 4	.0260076	-.391612	509.0466725		71

Notes: • Estimated standard error of $\hat{\alpha}_{15}$ = 0.0221141.
• Estimated standard error of $\hat{\alpha}_{21}$ = 0.0618044.

11.6.2 Model with Serial Correlation

The linear model was also solved and estimated for the case where u_{1t} in (11.17) follows a first-order autoregressive process, with $\rho_1 = .7$. Steps (i)–(v) were used with a damping factor of .25 to solve for the first observation, with steps 1–4 used thereafter. Some initial experimentation with no damping factor for calculating the initial condition indicated that convergence would either not be achieved or would be very slow. Again, for the same set of coefficient values, the same likelihood value was obtained using both the factorization technique and the method in Section 11.2. A summary of the calculations for the method is presented in the lower section of Table 11-1. The required number of passes in this case was about 37,563, which is about 35 percent greater than the number required for the model without serial correlation.

An attempt was made to use the less expensive method to estimate this version of the model, but this was not successful. The expectations did not appear to be well-behaved functions of the coefficients, and quite different derivatives were obtained for different step sizes. The values of L computed using the derivatives were generally not very close to the values of L computed by solving the entire model. It appears for this version that the entire model has to be solved for each new evaluation of L.

The use of the less expensive method for the small linear model thus produced mixed results. More estimation of alternative models is needed

before one can determine whether the difficulties with the serial correlation case are specific to the example and, if so, whether the example is representative of the type of model that is likely to be estimated in practice.

11.7 Results for the US Model with Rational Expectations in the Bond and Stock Markets (USRE1 and USRE2)

An interesting exercise with the US model is to consider how its policy properties would differ if it were specified to be consistent with the assumption of rational expectations in the bond and stock markets. The method in Section 11.2 can be used to solve the model in this case. The modifications of the model to incorporate the rational expectations assumption are discussed first, and then the policy properties of the different versions are compared.

11.7.1 The Two Term Structure Equations

The two term structure equations in the model, Eqs. 23 and 24, are discussed in Section 4.1.6. In each equation the long-term rate, *RB* or *RM*, is a function of current and lagged values of the short-term rate, *RS*. The theory on which these equations are based is the expectations theory of the term structure of interest rates. According to this theory, the return from holding an n-period security is equal to the expected return from holding a series of one-period securities over the n periods. Let RS^e_{t+i} denote the expected one-period rate of return for period $t+i$, the expectation being conditional on information available as of the beginning of period t, and let R_t denote the yield to maturity in period t on an n-period security. Then according to the expectations theory,

$$(11.19) \quad (1+R_t)^n = (1+RS^e_t)(1+RS^e_{t+1}) \ldots (1+RS^e_{t+n-1}).$$

When considered by themselves, Eqs. 23 and 24 are consistent with the expectations theory in the sense that the current and lagged values of *RS* are proxies for the expected future values in (11.19). When these equations are considered as part of the overall model, however, they are not consistent with the expectations theory *if* expectations of the future values of *RS* are rational. The reason for this is that in simulations of the model, the predicted values of the long-term rates and the short-term rates do not in general satisfy (11.19).

The US model can be modified to be consistent with the rational expectations assumption by dropping Eqs. 23 and 24 from the model and requiring instead that the solution values of *RS*, *RB*, and *RM* satisfy (11.19), where R_t in (11.19) represents *RB* and *RM*. The resulting model, which will be called

USRE1, is then consistent with the assumption of rational expectations in the bond market *if* (1) people believe that USRE1 is the true model and know how to solve it and (2) people at any one time have the same set of forecasts regarding the future values of the exogenous variables and the same set of expectations regarding the future values of the error terms. Given these assumptions, the solution values of the endogenous variables are people's expectations of these values (ignoring the bias due to the nonlinearity of the model). Since three of the endogenous variables in the model are *RS*, *RB*, and *RM*, if the solution values of these variables satisfy (11.19), then people's expectations are consistent with this equation.

11.7.2 The Stock-Price Equation

The stock-price or capital-gains equation, Eq. 25, is also discussed in Section 4.1.6. The capital-gains variable, *CG*, is a function of the change in *RB*, the change in after-tax cash flow, and the one-quarter-lagged value of the change in after-tax cash flow. The theory on which this equation is based is that the value of stocks is the present discounted value of expected future after-tax cash flow, the discount rates being the expected future short-term interest rates. Let $\pi_t = CF_t - T_{fgt} - T_{fst}$ denote the actual value of after-tax cash flow for period t, and let π^e_{t+i} denote the expected value for period $t+i$, the expectation being conditional on information available as of the beginning of period t. Let SP_t denote the value of stocks for period t based on information as of the beginning of period t. Then according to the theory

$$(11.20)\quad SP_t = \frac{\pi^e_t}{1+RS^e_t} + \frac{\pi^e_{t+1}}{(1+RS^e_t)(1+RS^e_{t+1})} + \cdots + \frac{\pi^e_{t+T}}{(1+RS^e_t)(1+RS^e_{t+1})\cdots(1+R^e_{t+T})},$$

where T is large enough to make the last term in (11.20) negligible. By definition

$$(11.21)\quad CG_t = SP_t - SP_{t-1},$$

where *CG* is the capital-gains variable used on the LHS of Eq. 25.

When considered by itself, Eq. 25 is consistent with (11.20) and (11.21) in the sense that the change in the bond rate is a proxy for expected future interest rate changes and the changes in after-tax cash flow are proxies for expected future changes. When considered as part of the overall model, Eq.

25 is not consistent with (11.20) and (11.21) if expectations of the future values are rational; this is because in simulations of the model the predicted values of CG do not in general satisfy (11.20)–(11.21).

The US model can also be modifid to be consistent with the rational expectations assumption regarding stock prices by dropping Eq. 25 and requiring instead that the solution values of CG satisfy (11.20)–(11.21). If this modification is made in conjunction with the modification regarding the term structure of interest rates, the resulting model, which will be called USRE2, is consistent with the assumption of rational expectations in both the bond and stock markets. Note in this case that because RS is used as the discount rate in (11.20), the expected return on stocks is the same as the expected return on bonds. There are no arbitrage opportunities in USRE2 between bonds and stocks, just as there are none in either USRE1 or USRE2 between bonds of different maturities.

RS is in units of percentage points at an annual rate, and for use in (11.19) and (11.20) in the following experiments, each RS term was divided by 400. This puts RS in units of percent at a quarterly rate.

11.7.3 The Policy Experiments

Unanticipated Change

Since both USRE1 and USRE2 have expected future values on the RHS of some equations, the solution method in Section 11.2 must be used to solve the models. Before they can be solved, however, some assumption must be made about n in (11.19) and T in (11.20). For present purposes both n and T were taken to be 32 quarters. The policy experiment consisted of a permanent increase in C_g (from its historical values) of 1.0 percent of real GNP. This is the same experiment as the first experiment in Table 9-1 except for a different period. The period here is 1958I–1960IV; this early period was chosen so that enough future data would be available to avoid having to make any assumptions about values of variables beyond the end of the data.

The value of h in (11.1) for both models is 31. The initial value of k in step 1 was chosen to be 67. This required initial guesses of the expectations of the future values of RS and of after-tax cash flow for 1958I through 1990II, although the values for the last 31 quarters are not changed during the solution process. For all but the last 31 quarters the initial expected values were taken to be the actual values. For the last 31 quarters (1982IV–1990II) the values were taken to be the 1982III values.

An important question for the experiment is how to handle the fact that (11.19) and (11.20)–(11.21) do not fit the data perfectly. The present experiment is not meant to be a test of the assumption of rational expectations in the bond and stock markets, but merely to examine the sensitivity of the properties of the model to this assumption. Given this, the easiest thing to do is to add error terms to (11.19) (for both *RB* and *RM*) and to (11.20) in such a way that the equations fit perfectly when the expected values are taken to be the actual values. If the actual values of the error terms are also used for the other equations, the solution of the model using the actual values of the exogenous variables (including C_g) is the perfect tracking solution. The base values for the C_g experiment are thus the actual values, which is the same as for the experiments in Chapter 9. The actual values of the exogenous variables were used for the experiment.

The error terms in (11.19) and (11.20) are not assumed to be serially correlated, which means that steps (a)–(e) in Section 11.2.2 do not have to be used. Even though some of the stochastic equations in the model have serially correlated errors, steps (a)–(e) do not have to be used unless the serial correlation occurs in equations with explanatory expectations variables.

The estimated policy effects are presented in Table 11-3. The solution method in Section 11.2 worked quite well in solving USRE1 and USRE2. For USRE2, for example, the number of type II iterations required for convergence was 28 for $k = 67$. When k was increased by one, the required number was 17. Type III convergence was achieved at this point. In other words, the initial value of k was chosen large enough so that increasing it by one more had negligible effects on the solution values for the first 32 quarters. For $h = 31$ and $k = 67$, each type II iteration requires solving the model for $31 + 67 + 1 = 99$ quarters. The solution for each quarter requires about .2 seconds on the IBM 4341, so the solution time for one type II iteration is about 19.8 seconds. The total time for the 28 type II iterations was thus about 9.2 minutes. For k increased by one, the time per type II iteration is only .2 seconds longer. The time for the other 17 type II iterations was thus about 20 seconds $\times$ 17 = 5.7 minutes. The total time required for the solution for USRE2 was thus about 14.9 minutes. The times for USRE1 were similar. If one compares these times to the time required to solve the regular version of the US model for the 12 quarters in Table 11-3 of 12 $\times$ .2 seconds = 2.4 seconds, the USRE1 and USRE2 models are about 373 times more expensive to solve than the US model.

It is important to note with respect to solution times that the model only had to be solved once for each set of results in Table 11-3. The reason for this

TABLE 11-3. Estimated effects of an unanticipated increase in C_g for US, USRE1, and USRE2

	1958				1959				1960				Sum[a]
	I	II	III	IV	I	II	III	IV	I	II	III	IV	
GNPR: Real GNP													
US	1.12	1.34	1.34	1.27	1.15	1.02	.88	.77	.67	.60	.54	.50	4.95
USRE1	.99	1.04	.97	.90	.84	.78	.74	.72	.71	.71	.70	.69	4.34
USRE2	1.00	1.04	.96	.89	.83	.77	.73	.70	.69	.67	.66	.64	4.25
GNPD: GNP deflator													
US	.04	.19	.26.	.33	.42	.47	.50	.53	.54	.55	.56	.56	
USRE1	.01	.12	.16	.20	.27	.31	.35	.38	.41	.45	.47	.49	
USRE2	.02	.12	.16	.20	.28	.32	.36	.39	.41	.44	.47	.49	
100·UR: Unemployment rate (percentage points)													
US	-.06	-.11	-.12	-.11	-.10	-.10	-.08	-.07	-.06	-.05	-.05	-.05	
USRE1	-.05	-.08	-.09	-.08	-.07	-.07	-.06	-.06	-.06	-.06	-.06	-.07	
USRE2	-.05	-.08	-.08	-.07	-.07	-.06	-.06	-.06	-.06	-.06	-.06	-.06	
RS: Bill rate (percentage points)													
US	.07	.10	.11	.12	.13	.13	.13	.13	.12	.12	.11	.11	
USRE1	.06	.07	.08	.09	.10	.10	.11	.11	.11	.11	.12	.12	
USRE2	.06	.07	.08	.09	.10	.10	.10	.11	.11	.11	.11	.11	
RB: Bond rate (percentage points)													
US	.02	.03	.04	.05	.06	.07	.08	.09	.10	.10	.10	.11	
USRE1	.11	.12	.12	.12	.12	.12	.12	.13	.13	.13	.14	.14	
USRE2	.10	.10	.10	.11	.11	.11	.11	.11	.11	.11	.12	.12	
RM: Mortgage rate (percentage points)													
US	.01	.03	.05	.06	.08	.09	.10	.10	.11	.11	.12	.12	
USRE1	.10	.10	.10	.11	.11	.11	.11	.11	.11	.12	.12	.12	
USRE2	.09	.09	.09	.09	.10	.10	.10	.10	.10	.10	.10	.10	
CG: Capital gains variable (billions of current dollars)													
US	-.11	-.06	-.68	-.45	-.34	-.30	-.26	-.20	-.18	-.14	-.12	-.09	
USRE1	-2.50	-.20	-.52	-.21	-.14	-.06	-.08	-.07	-.10	-.13	-.16	-.14	
USRE2	-3.40	-.26	-.16	-.15	-.15	-.15	-.17	-.18	-.19	-.17	-.17	-.17	
CF: Cash flow (billions of current dollars)													
US	.18	.11	.07	.04	.02	-.01	-.03	-.05	-.07	-.08	-.09	-.10	
USRE1	.11	-.00	-.07	-.10	-.12	-.13	-.15	-.16	-.17	-.18	-.19	-.20	
USRE2	.12	.01	-.05	-.08	-.09	-.11	-.12	-.13	-.13	-.14	-.15	-.15	
INT_f: Interest payments of the firm sector (billions of current dollars)													
US	.01	.02	.03	.04	.06	.08	.10	.12	.13	.15	.16	.17	
USRE1	.05	.09	.12	.15	.17	.19	.21	.22	.24	.25	.26	.28	
USRE2	.05	.08	.11	.13	.15	.17	.18	.19	.20	.21	.22	.24	

Notes: a. Sum of the changes (at quarterly rates) over the 12 quarters, in billions of 1972 dollars.
- C_g was increased by 1.0 percent of GNPR beginning in 1958 I (sustained increase).
- The changes for GNPR and GNPD are in percentage points.

is that the actual values of the exogenous variables were used and that the length of the simulation period of interest (12 quarters) was less than h. (See the discussion at the end of Section 11.2.1 for an explanation of this.)

The results in Table 11-3 are fairly easy to understand. For all three versions, the Fed responded to the increase in C_g by raising RS. In the regular version this had a gradual effect over time on RB and RM through the term structure equations. In the other two versions, however, knowledge that the

Fed was going to raise *RS* in the future was incorporated immediately into the long-term rates, and therefore the initial changes in *RB* and *RM* were greater for USRE1 and USRE2 than for the US model. This led to lower initial increases in real GNP and to smaller initial decreases in the unemployment rate. The lower initial increases in real GNP led to smaller increases in the GNP deflator.

Because of the lower initial increases in real GNP for USRE1 and USRE2, the initial increases in *RS* were also lower. In other words, the Fed responded less with respect to increasing *RS* in these two cases. The higher initial values of *RB* and *RM* for USRE1 and USRE2 required less of an increase in *RS* in order to lessen the expansionary impact of the increase in C_g.

One puzzling feature of the results in Table 11-3 is why the initial change in stock prices (*CG*) is negative for USRE2. It is more negative for USRE2 than it is for USRE1, which through the wealth effects in the model leads to a slightly more expansionary economy for USRE1 than for USRE2. If future values of cash flow are higher because of the expansion, this information should be reflected immediately in higher stock prices for USRE2. There are, of course, two effects on stock prices, a positive one through higher future values of cash flow and a negative one through higher future values of the discount rates. It may merely be that the negative discount rate effect dominates for USRE2. This is not, however, the case. The problem is that future values of cash flow are smaller rather than larger. (This can be seen for the first 12 quarters in Table 11-3.) The reason for this is that interest payments of the firm sector, which are subtracted from cash flow, are larger because of the higher bond rate. (This can also be seen for the first 12 quarters in Table 11-3).

The puzzling result is thus due to the higher interest payments of the firm sector. Interest payments are determined by Eq. 19 in the model. This equation, as discussed in Section 4.1.5, does not have good statistical properties, and in particular it may be that the bond rate coefficient in the equation is too large. The USRE1 versus USRE2 results thus unfortunately depend on a questionable equation. In order to see how sensitive the results in Table 11-3 are to the interest payments equation, the experiments were done over with the interest payments equation dropped and interest payments taken to be exogenous. The results of these experiments are presented in Table 11-4. The results for US and USRE1 are not much affected, but it is now the case that future values of cash flow are positive. The initial change in stock prices for USRE2 is now positive. *CG* increased by 1.28 in the first quarter for USRE2, whereas it decreased by 1.96 for USRE1. The decrease for USRE1 is a result

TABLE 11-4. Estimated effects of an unanticipated increase in C_g for US, USRE1, and USRE2 with INT_f exogenous

	1958				1959				1960				Sum[a]
	I	II	III	IV	I	II	III	IV	I	II	III	IV	
GNPR: Real GNP													
US	1.12	1.33	1.33	1.25	1.12	.98	.83	.70	.60	.52	.45	.40	4.69
USRE1	1.00	1.07	1.00	.93	.86	.79	.73	.69	.67	.65	.63	.60	4.27
USRE2	.97	1.09	1.07	1.04	1.00	.96	.92	.89	.88	.86	.84	.83	5.05
GNPD: GNP deflator													
US	.04	.18	.26	.32	.41	.46	.49	.51	.52	.53	.53	.53	
USRE1	.02	.13	.17	.22	.30	.34	.38	.41	.43	.46	.48	.50	
USRE2	.01	.12	.17	.23	.32	.37	.43	.47	.51	.55	.58	.61	
100 UR: Unemployment rate (percentage points)													
US	-.06	-.10	-.12	-.11	-.10	-.09	-.07	-.06	-.05	-.04	-.04	-.03	
USRE1	-.05	-.08	-.09	-.08	-.07	-.07	-.06	-.05	-.05	-.05	-.05	-.05	
USRE2	-.05	-.09	-.10	-.10	-.09	-.09	-.09	-.09	-.09	-.09	-.09	-.08	
RS: Bill rate (percentage points)													
US	.07	.10	.11	.12	.13	.13	.12	.12	.11	.11	.10	.10	
USRE1	.06	.08	.08	.09	.10	.10	.10	.10	.10	.11	.11	.10	
USRE2	.06	.08	.09	.10	.11	.11	.12	.12	.13	.13	.13	.13	
RB: Bond rate (percentage points)													
US	.02	.03	.04	.05	.06	.07	.08	.09	.09	.10	.10	.10	
USRE1	.09	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	
USRE2	.12	.12	.12	.12	.12	.12	.12	.12	.12	.12	.13	.13	
RM: Mortgage rate (percentage points)													
US	.01	.03	.05	.06	.07	.09	.09	.10	.10	.11	.11	.11	
USRE1	.08	.08	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	
USRE2	.10	.10	.11	.11	.11	.11	.11	.11	.11	.11	.11	.11	
CG: Capital gains variable (billions of current dollars)													
US	-.10	-.03	-.64	-.39	-.26	-.21	-.17	-.11	-.09	-.05	-.03	-.00	
USRE1	-1.96	.05	-.33	-.06	.01	.06	.04	.04	.01	-.01	-.03	-.01	
USRE2	1.28	-.00	.07	.09	.10	.10	.09	.09	.10	.10	.09	.08	
CF: Cash flow (billions of current dollars)													
US	.19	.13	.10	.08	.07	.06	.06	.06	.06	.06	.07	.07	
USRE1	.17	.09	.06	.05	.06	.06	.06	.07	.07	.08	.08	.08	
USRE2	.16	.10	.07	.06	.07	.07	.07	.08	.08	.08	.09	.09	
INT_f: Interest payments of the firm sector (billions of current dollars)													
US	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
USRE1	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	
USRE2	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	

Notes: a. Sum of the changes (at quarterly rates) over the 12 quarters, in billions of 1972 dollars.
- C_g was increased by 1.0 percent of GNPR beginning in 1958 I (sustained increase).
- The changes for GNPD and GNPR are in percentage points.

of the higher value of *RB*, which appears as an explanatory variable in the *CG* equation. The economy is now more expansionary for USRE2 than it is for USRE1.

This feature of the results regarding the difference between USRE1 and USRE2 is thus sensitive to the interest payments equation. The results in Tables 11-3 and 11-4 bound the differences in the sense that interest payments are probably too sensitive to interest rates in Table 11-3 and not

sensitive enough in Table 11-4. For purposes of illustrating the properties of the two versions of the US model, these results are sufficient.

An interesting aspect of the results is that the sums of the GNP changes across the 12 quarters are quite close. The timing of the GNP changes differs between the US model and the two rational expectations versions, but this is to some extent the only substantial difference among the results.

Anticipated Change

The experiment just reported is an unanticipated increase in C_g beginning in 1958I. If the increase had been announced before this time, the quarters prior to the enactment would have been affected in models USRE1 and USRE2. To investigate this, a second experiment was run in which it was assumed that the announcement of the C_g increase beginning in 1958I was made at the beginning of 1956I. The results of this experiment are reported in Table 11-5. (The interest payments equation was used for these results.) The initial value of k was taken to be 75 for this experiment rather than 67, and the starting quarter was 1956I rather than 1958I. Otherwise, the procedure for this experiment was the same as that for the first. Convergence was achieved in two type III iterations for each model, and the solution times were similar to those for the first experiment.

The results for the US model in Table 11-5 are the same as those in Table 11-3. The announcement has no effect on this model since it is not forward-looking. For the other two models, knowledge that the Fed will raise *RS* in the future gets incorporated immediately into *RB* and *RM*, which has a negative effect on real output. Real GNP is lower in 1956 and 1957 because of the higher long-term interest rates. *RS* is lower in these two years because of the contractionary economy; *RS* begins to rise after the increase in C_g actually takes place.

The sum of the output changes across the 20 quarters is 4.95 for the US model, 3.96 for USRE1, and 3.44 for USRE2. The difference between the US model and the others is larger here than it is in the first experiment, which is due to the negative effects in the first two years for USRE1 and USRE2. The reason the economy is less expansionary for USRE2 than for USRE1 is again because of the interest payments equation. The opposite result would be obtained if the interest payments equation were dropped.

Conclusions

These experiments give a good indication of the sensitivity of the policy properties of the model to the assumption of rational expectations in the bond

TABLE 11-5. Estimated effects of an anticipated increase in C_g for US, USRE1, and USRE2

	1956				1957				1958				1959				1960				Sum[a]
	I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV	I	II	III	IV	
GNPR: Real GNP																					
US	0.	0.	0.	0.	0.	0.	0.	0.	1.12	1.34	1.34	1.27	1.15	1.02	.88	.77	.67	.60	.54	.50	4.95
USRE1	-.12	-.30	-.43	-.52	-.56	-.57	-.57	-.56	.58	.89	1.01	1.08	1.10	1.10	1.10	1.09	1.09	1.09	1.08	1.07	3.96
USRE2	-.09	-.29	-.44	-.54	-.59	-.61	-.61	-.61	.53	.83	.94	1.00	1.03	1.03	1.02	1.01	1.00	.99	.97	.94	3.44
GNPD: GNP deflator																					
US	0.	0.	0.	0.	0.	0.	0.	0.	.04	.19	.26	.33	.42	.47	.50	.53	.54	.55	.56	.56	
USRE1	-.03	-.08	-.13	-.19	-.24	-.28	-.32	-.35	-.33	-.20	-.13	-.06	.05	.13	.21	.28	.35	.42	.47	.52	
USRE2	-.02	-.07	-.12	-.18	-.23	-.28	-.32	-.35	-.34	-.21	-.15	-.07	.03	.11	.18	.24	.31	.37	.42	.46	
100·UR: Unemployment rate (percentage points)																					
US	0.	0.	0.	0.	0.	0.	0.	0.	-.06	-.11	-.12	-.11	-.10	-.10	-.08	-.07	-.06	-.05	-.05	-.05	
USRE1	.01	.02	.04	.05	.06	.06	.06	.06	-.01	-.06	-.09	-.10	-.10	-.10	-.10	-.10	-.11	-.11	-.11	-.11	
USRE2	.01	.02	.04	.06	.06	.07	.07	.07	.00	-.05	-.07	-.08	-.08	-.09	-.09	-.09	-.09	-.09	-.09	-.09	
RS: Bill rate (percentage points)																					
US	0.	0.	0.	0.	0.	0.	0.	0.	.07	.10	.11	.12	.13	.13	.13	.13	.12	.12	.11	.11	
USRE1	-.01	-.02	-.03	-.04	-.05	-.06	-.06	-.07	.01	.03	.06	.08	.10	.11	.12	.13	.14	.15	.15	.16	
USRE2	-.01	-.02	-.03	-.05	-.06	-.06	-.07	-.07	-.00	.03	.05	.07	.09	.10	.11	.12	.13	.13	.14	.14	
RB: Bond rate (percentage points)																					
US	0.	0.	0.	0.	0.	0.	0.	0.	.02	.03	.04	.05	.06	.07	.08	.09	.10	.10	.10	.11	
USRE1	.09	.09	.10	.10	.11	.11	.12	.13	.14	.15	.15	.15	.16	.16	.16	.16	.17	.17	.18	.18	
USRE2	.07	.07	.08	.08	.09	.09	.10	.10	.11	.12	.12	.12	.12	.13	.13	.13	.13	.13	.14	.14	
RM: Mortgage rate (percentage points)																					
US	0.	0.	0.	0.	0.	0.	0.	0.	.01	.03	.05	.06	.08	.09	.10	.10	.11	.11	.12	.12	
USRE1	.08	.08	.09	.09	.10	.10	.11	.11	.12	.13	.13	.14	.14	.14	.15	.15	.15	.15	.16	.16	
USRE2	.06	.07	.07	.07	.08	.08	.09	.09	.10	.10	.11	.11	.11	.11	.11	.11	.11	.12	.12	.12	
CG: Capital gains variable (billions of current dollars)																					
US	0.	0.	0.	0.	0.	0.	0.	0.	-.11	-.06	-.68	-.45	-.34	-.30	-.26	-.20	-.18	-.14	-.12	-.09	
USRE1	-2.27	-.35	-.29	-.22	-.26	-.26	-.22	-.31	-.02	-.09	-.48	-.24	-.19	-.10	-.11	-.10	-.13	-.17	-.20	-.17	
USRE2	-3.71	-.21	-.20	-.20	-.20	-.21	-.21	-.22	-.22	-.28	-.18	-.18	-.18	-.18	-.21	-.22	-.23	-.21	-.20	-.20	
CF: Cash flow (billions of current dollars)																					
US	0.	0.	0.	0.	0.	0.	0.	0.	.18	.11	.07	.04	.02	-.01	-.03	-.05	-.07	-.08	-.09	-.10	
USRE1	-.06	-.12	-.15	-.18	-.21	-.23	-.25	-.27	-.11	-.18	-.23	-.26	-.27	-.29	-.31	-.32	-.33	-.35	-.36	-.38	
USRE2	-.05	-.10	-.13	-.15	-.17	-.19	-.21	-.23	-.06	-.12	-.16	-.19	-.20	-.21	-.22	-.23	-.24	-.25	-.26	-.27	
INT_f: Interest payments of the firm sector (billions of current dollars)																					
US	.0	.0	.0	.0	.0	.0	.0	.0	.01	.02	.03	.04	.06	.08	.10	.12	.13	.15	.16	.17	
USRE1	.04	.08	.11	.13	.16	.18	.20	.22	.25	.27	.29	.31	.33	.35	.37	.39	.40	.42	.45	.47	
USRE2	.03	.06	.08	.10	.12	.14	.16	.18	.19	.21	.23	.24	.26	.27	.29	.30	.31	.33	.34	.35	

Notes: a. Sum of the changes (at quarterly rates) over the 20 quarters, in billions of 1972 dollars.
- C_g was increased by 1.0 percent of GNPR beginning in 1958 I (sustained increase). The increase was announced in 1956 I.
- The changes for GNPR and GNPD are in percentage points.

and stock markets. It is clear that there are some important quantitative policy differences, especially with respect to timing and anticipated changes. The rational expectations assumption is clearly of some quantitative importance.

It should be stressed again that the results in this section provide no tests of the rational expectations assumption. For purposes of the experiments, (11.19) and (11.20) have been made to fit perfectly by merely adding the actual errors to them before they are solved. These errors are in fact quite large relative to the errors in the estimated term structure and capital gains equations. This is not, however, evidence against the specification of (11.19) and (11.20). Some of the reasons for this are the following.

1. The *RB* and *RM* rates are not eight-year rates, as assumed here, and therefore a closer matching of the rate data to n would be needed in any tests.
2. The value of T used for (11.20), 32 quarters, is not large enough to make the last term in the equation negligible.
3. The data on cash flow after taxes and stock prices do not match exactly.
4. The use of actual values of *RS* and π for the expected future values in the construction of the error terms for (11.19) and (11.20) is not appropriate.

None of these problems are important for the sensitivity experiments performed in this section, but they are obviously so for testing. If better data were collected so that 1 and 3 were taken care of and if a larger value of T were used so that 2 was taken care of, then the rational expectations assumption with respect to the bond and stock markets could be tested by, say, comparing the accuracy of the predictions from USRE2 and US, especially the predictions of *RB*, *RM*, and *CG*. For USRE2 one would have to choose for each beginning quarter of a prediction period a set of future values of the exogenous variables that one believes were expected at the time. The predictions for each different beginning quarter would be based on a different set of future values of the exogenous variables. The joint hypothesis that would be tested by this procedure is that (a) people know USRE2 and believe it to be true, including (11.19) and (11.20); (b) the chosen exogenous variable values and error terms correctly reflect the expectations at the time; and (c) expectations with respect to future values of *RS* and cash flow after taxes are rational.

11.8 Results for Sargent's Model (SARUS)

The estimation of Sargent's model is somewhat involved, as is true of any rational expectations model, and it will be easiest to discuss the estimation of

TABLE 11-6. Coefficient estimates of Sargent's model for 1954 II - 1982 III

Equation (1): Un_t is the LHS variable

	constant	t	$p_t - E_{t-1}p_t$	Un_{t-1}	Un_{t-2}	Un_{t-3}	Un_{t-4}	SE
2SLS	.00169	.0000232	—	1.635	-.800	.005	.105	.00304
	(1.32)	(2.25)		(18.09)	(4.52)	(0.03)	(1.16)	
FIML	.00192	.0000245	—	1.648	-.806	.016	.082	
	(1.39)	(2.30)		(17.76)	(4.34)	(0.09)	(0.90)	
FIML	-.00007	.0000173	-.380	1.633	-.767	.053	.066	

Equation (2): nf_t is the LHS variable

	constant	t	$p_t - E_{t-1}p_t$	Un_t	nf_{t-1}	nf_{t-2}	nf_{t-3}	nf_{t-4}	SE
2SLS	-.0033	.0000336	—	-.047	.945	.057	-.059	.048	.00330
	(0.22)	(1.82)		(1.58)	(10.39)	(0.46)	(0.47)	(0.51)	
FIML	-.0366	.0000614	—	-.002	.963	.041	-.056	-.014	
	(2.70)	(3.30)		(0.07)	(10.65)	(0.34)	(0.50)	(0.18)	
FIML	-.0393	.0000660	.207	-.015	.960	.036	-.057	-.012	

Equation (3): y_t is the LHS variable

	constant	t	n_t	n_{t-1}	n_{t-2}	n_{t-3}	n_{t-4}	$\hat{\rho}_1$	$\hat{\rho}_2$	SE
2SLS	3.04	.00123	1.833	-.466	-.225	-.289	.054	.930	.054	.00774
	(2.26)	(0.29)	(7.28)	(1.99)	(1.19)	(1.56)	(0.33)	(8.39)	(0.49)	
FIML	.98	.00024	2.147	-.539	-.186	-.199	.147	.870	.096	
	(0.80)	(0.11)	(5.16)	(1.73)	(1.06)	(1.15)	(1.08)	(9.17)	(1.00)	
FIML	1.08	.00022	2.190	-.619	-.188	-.156	.123	.872	.099	

Equation (5c): $m_t - p_t$ is the LHS variable

	constant	t	R_t	R_{t-1}	$R_{t-2} \cdots R_{t-7}$	y_t	y_{t-1}	$y_{t-2} \cdots y_{t-7}$	$\hat{\rho}_1$	$\hat{\rho}_2$	SE
2SLS	-.282	.0043	.0078	.0037	-.0005	-.667	.189	-.091	.841	.070	.00834
	(0.20)	(1.82)	(2.67)	(1.08)	(0.12)	(4.43)	(1.67)	(0.90)	(7.88)	(0.66)	
FIML	-3.623	-.0009	.0058	.0066	.0026	-.122	.039	-.088	.890	.072	
	(1.46)	(0.24)	(2.01)	(2.08)	(0.71)	(0.37)	(0.30)	(0.92)	(8.95)	(0.72)	
FIML	-2.747	-.0009	.0079	.0075	.0062	-.447	.061	-.098	.895	.060	

Equation (4): R_t is the LHS variable

	constant	t	R_{t-1}	R_{t-2}	R_{t-3}	R_{t-4}	SE
OLS	.105	.00643	1.269	-.518	.632	-.460	.289
	(1.46)	(2.77)	(14.53)	(3.86)	(4.62)	(4.82)	

Equation (6): $n_t = nf_t - Un_t + pop_t$

Notes: • FSRs for 2SLS, all equations: constant, t, Un_{t-1}, Un_{t-2}, Un_{t-3}, Un_{t-4}, nf_{t-1}, nf_{t-2}, nf_{t-3}, nf_{t-4}, n_{t-1}, n_{t-2}, n_{t-3}, n_{t-4}, n_{t-5}, n_{t-6}, R_t, R_{t-1}, ..., R_{t-9}, y_{t-1}, y_{t-2}, ..., y_{t-9}, pop_t, m_t, $m_{t-1} - p_{t-1}$, $m_{t-2} - p_{t-2}$.

• 2SLS for equation (1) is OLS because there are no RHS endogenous variables.

it in steps. The model is presented in Section 5.4, and the reader should review this material before reading this section, in particular the material in Tables 5-3 and 5-4. The model consists of five stochastic equations and one identity. These equations are listed in Table 11-6. The first thing to remember about the model is that the error term in Eq. (4) is assumed to be uncorrelated with the other error terms in the model. This means that Eq. (4) can be treated separately from the rest and simply estimated by OLS.

The key variable in Sargent's model is $p_t - E_{t-1}p_t$, which is an explanatory variable in Eqs. (1) and (2). Without this variable, the model is not a rational expectations model and can thus be estimated by standard techniques. The first step in the estimation work was to estimate the model by 2SLS without the $p_t - E_{t-1}p_t$ variable included. These estimates are presented first in Table 11-6. The first-stage regressors that were used for these estimates are listed at the bottom of the table. The next step was to estimate this same version of the model by FIML. These estimates are presented next in Table 11-6. The 2SLS estimates were used as starting values. The value of L (see Eq. 6.33) at the starting point was 2438.49. The Parke algorithm was allowed to run for 40 iterations, which increased L by 10.37 to 2448.86. Near the end of the 40 iterations, L was increasing by about .01 per iteration. Each iteration corresponds to about 180 function evaluations and takes about 65 seconds on the IBM 4341. At the stopping point the covariance matrix of the coefficient estimates was computed ($\hat{V}_4$ in Eq. 6.34), and this is where the t-statistics for the first set of FIML estimates in Table 11-6 come from.

The next step was to add the expectations variable to the model and estimate it using the method in Section 11.2. The solution of the model is fairly easy because there are no expectations variables for periods $t + 1$ and beyond, only for period t. This means that no type II or type III iterations have to be performed. In order to get the values for $E_{t-1}p_t$ ($t = 1, \ldots, T$) that are needed for the computation of L, the model is simply solved each period using the expected values of the exogenous variables. The predicted values of p_t from this solution are the values used for $E_{t-1}p_t$. For purposes of estimation there are three exogenous variables: m_t, pop_t, and R_t. As noted in Table 5-4, the expected values of m_t and pop_t were taken to be predicted values from eighth-order autoregressive equations. The expected values of R_t were taken to be the predicted values from Eq. (4).

In this third step each evaluation of L requires that the model be solved for each of the 114 observations of the sample period. This solution takes about 10.5 seconds on the IBM 4341. As noted earlier, the number of function evaluations required per iteration of the Parke algorithm is about 180, which

takes about 65 seconds for the nonrational expectations version of the model. The total time per iteration for the complete model is thus about 10.5 seconds $\times$ 180 + 65 seconds = 32.6 minutes. Because of the cost per iteration, the Parke algorithm was only allowed to run for eight iterations. The FIML estimates of the nonrational expectations version were used as starting values. The value of L was increased from the starting value of 2448.86 to 2475.16, which is a change of 26.30 points.

The set of estimates at this point is the third set presented in Table 11-6. The key result in this table is that both coefficient estimates for $p_t - E_{t-1}p_t$ are of the expected sign (negative in Eq. 1 and positive in Eq. 2). According to the theory behind the model, positive price surprises should lead to a fall in the unemployment rate and a rise in labor supply, and the results are consistent with this theory.

The covariance matrix of the third set of coefficient estimates was not computed because of the expense, but it is the case that the two coefficient estimates for $p_t - E_{t-1}p_t$ are jointly significant. This can be seen by performing a likelihood ratio test. Let L^* denote the optimal value of L and let L^{**} denote the value of L obtained by maximizing the likelihood function subject to the constraint that both coefficients are zero. Then $2(L^* - L^{**})$ has an asymptotic χ^2 distribution with two degrees of freedom. The value of L^{**} is 2448.86 from the above results. A lower bound for the value of L^* is the final value of 2475.16. (This is only a lower bound because the Parke algorithm was not allowed to run long enough to obtain the maximum.) Twice the difference between the lower bound for L^* and L^{**} is 52.60, which is clearly greater than the critical χ^2 value at the 95-percent confidence level of 5.99. Therefore, even using this conservative value, the two coefficient estimates are highly significant.

Because of the expense of estimating Sargent's model, it was not feasible to use the method in Chapter 8 to examine the accuracy of the model. It did seem worthwhile, however, to try to get a rough idea of its accuracy. This was done by computing within-sample root mean squared errors (RMSEs). RMSEs were computed for one- through eight-quarter-ahead predictions for the 1970I–1982III period. This was done for the three estimates of Sargent's model in Table 11-6, for the ARUS model, and for the US model. The results are presented in Table 11-7. The results in Table 11-7 for the US model are the same as those in Table 8-5 (2SLS estimates).

Before discussing the results in Table 11-7, one should be clear about how the rational expectations version of Sargent's model is solved when the simulation is dynamic. (The simulations that were used in the above estima-

TABLE 11-7. Root mean squared errors of within-sample forecasts for 1970 I - 1982 III for SARUS, ARUS, and US

	Number of quarters ahead							
	1	2	3	4	5	6	7	8
GNPR = e^y: Real GNP								
SARUS[a]	1.07	1.72	2.08	2.37	2.45	2.56	2.67	2.77
SARUS[b]	1.02	1.66	2.12	2.49	2.71	2.92	3.08	3.24
SARUS[c]	1.09	1.79	2.33	2.75	3.03	3.31	3.52	3.73
ARUS	1.04	1.60	2.05	2.37	2.56	2.73	2.85	3.01
US	.66	.81	1.08	1.25	1.43	1.61	1.73	1.81
GNPD = e^p: GNP deflator								
SARUS[a]	.75	1.04	1.25	1.50	1.83	2.02	2.29	2.56
SARUS[b]	1.00	1.35	1.71	2.12	2.59	2.88	3.22	3.59
SARUS[c]	1.09	1.54	1.93	2.35	2.89	3.18	3.48	3.83
ARUS	.38	.69	1.00	1.27	1.57	1.87	2.07	2.26
US	.44	.69	.88	1.05	1.18	1.23	1.25	1.22
100·UR = 100·Un: Unemployment rate (percentage points)								
SARUS[a]	.36	.68	.92	1.10	1.13	1.12	1.13	1.14
SARUS[b]	.33	.62	.84	1.04	1.12	1.16	1.19	1.22
SARUS[c]	.33	.63	.86	1.06	1.15	1.20	1.24	1.27
ARUS	.32	.61	.84	1.03	1.12	1.17	1.21	1.24
US	.29	.43	.55	.66	.75	.83	.90	.95

Notes: a. Second set of FIML estimates in Table 11-6 (complete model).
b. First set of FIML estimates in Table 11-6 ($p_t - E_{t-1}p_t$ excluded).
c. 2SLS estimates in Table 11-6.
* There are 51 observations for the one-quarter-ahead forecasts, 50 for the two-quarter-ahead forecasts, and so on.
* For the unemployment rate the errors are in the natural units of the variables. For real GNP and the GNP deflator the errors are percentage errors (in percentage points).

tion of the model were all static.) Remember that the model is actually solved two times per quarter to get the final solution values. The model is first solved using the expected values of the exogenous variables, which gives a solution value for $E_{t-1}p_t$. The model is then solved again using this solution value plus the actual values of the exogenous variables. For both the static and dynamic simulations the expected values of the two exogenous variables, m_t and pop_t, were taken to be static predictions from the two estimated eighth-order autoregressive equations. It would not be appropriate to use dynamic predictions for this purpose because of the exogeneity of m_t and pop_t themselves. For solution purposes, in contrast to estimation purposes, R_t is an endogenous variable, and therefore the above procedure for m_t and pop_t is not followed for R_t. The R_t equation, Eq. (4), is simply added to the model for solution purposes.

The results in Table 11-7 indicate that Sargent's model is not very accurate.

All three versions are considerably less accurate than the US model for all three variables. All versions are less accurate than the ARUS model for the GNP deflator. The three versions and ARUS are of about the same degree of accuracy for the unemployment rate. The rational expectations version of Sargent's model is slightly more accurate than ARUS for real GNP for the five- through eight-quarter-ahead predictions. The other two versions are less accurate than ARUS for real GNP. Although these results are subject to the reservations discussed in Chapter 8 regarding within-sample RMSE comparisons, they are clearly not encouraging regarding Sargent's model.

Sargent's model was the first serious attempt to construct an econometric version of the class of rational expectations models discussed in Section 3.1.7, and thus it is obviously very preliminary in nature. The negative results achieved here should thus be interpreted with some caution. It may be that with more work on models of this type, the accuracy will be much improved. It is really too early to judge this type of model. One discouraging feature about this work, however, is that there have been no attempts to follow up on Sargent's model or models like it. Unless more econometric work is done on this class of rational expectations models, it may lose by default.